Marzyfel

1/9 $\vec{v} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ en $\vec{w} = \begin{pmatrix} 1 \\ 3 \\ 2\sqrt{2} \end{pmatrix}$ in $\mathbb{R}^3$

Laat zien dat $W = \{ a\vec{v} + b\vec{w} \mid a, b \in \mathbb{R} \}$
een lineaire deelruimte is van $V = \mathbb{R}^3$

- als $a\vec{v} + b\vec{w}$ in W zit, dan ook
  $\lambda \cdot (a\vec{v} + b\vec{w}) = (\lambda a)\vec{v} + (\lambda b)\vec{w}$
- als $a\vec{v} + b\vec{w}$ en $a'\vec{v} + b'\vec{w}$ in W zitten,
  dan ook ~~(a+a') v~~ hun som
  $(a\vec{v} + b\vec{w}) + (a'\vec{v} + b'\vec{w}) = (a + a')\vec{v} + (b + b')\vec{w}$.

Bereken de lengte van $\vec{v}$ en $\vec{w}$, en de hoek
tussen $\vec{v}$ en $\vec{w}$

$\|v\|^2 = 3^2 + 4^2 = 5^2$, dus $\|v\| = 5$
$\|w\|^2 = 1^2 + 3^2 + (2\sqrt{2})^2 = 1 + 9 + 8 = 18$, $\|w\| = \sqrt{18} = 3\sqrt{2}$
als θ de hoek tussen $\vec{v}$ en $\vec{w}$ is, dan is

$$\cos\theta = \frac{\langle v, w\rangle}{\|v\|\|w\|} = \frac{3 \cdot 1 + 4 \cdot 3}{5 \cdot 3\sqrt{2}} = \frac{15}{15\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$$

Dus $\theta = \arccos(\frac{1}{2}\sqrt{2}) = \pi/4$ (45°)

Geef een orthogonale basis van W

Orthogonaliseer de basis $\vec{v}, \vec{w}$ ~>
$\vec{b}_0' = \vec{v}$
$\vec{b}_1' = \vec{w} - \frac{\langle b_0', \vec{w}\rangle}{\langle b_0', b_0'\rangle} b_0' = \vec{w} - \frac{\langle v, w\rangle}{\langle v, v\rangle}\vec{v}$

Dus $\vec{b}_0' = \vec{v} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$, $b_1' = w - \frac{15}{25}\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 2\sqrt{2} \end{pmatrix} - \frac{3}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -4/5 \\ 3/5 \\ 2\sqrt{2} \end{pmatrix}$

Mazgfer

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Wat is de projectie van $\vec{u} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$ op W?

Projectie $\vec{p} = \dfrac{\langle e_0', \vec{u} \rangle}{\langle e_0', e_0' \rangle} e_0' + \dfrac{\langle e_1', \vec{u} \rangle}{\langle e_1', e_1' \rangle} e_1'$

$$= 0 + \frac{18\sqrt{2}}{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(2\sqrt{2}\right)^2} \cdot \begin{pmatrix} -4/5 \\ 3/5 \\ 2\sqrt{2} \end{pmatrix}$$

$$= \frac{18\sqrt{2}}{9} \cdot \begin{pmatrix} -4/5 \\ 3/5 \\ 2\sqrt{2} \end{pmatrix} = \begin{pmatrix} -8/5\sqrt{2} \\ 6/5\sqrt{2} \\ 8 \end{pmatrix}$$

check: $u - p = \begin{pmatrix} 8/5\sqrt{2} \\ -6/5\sqrt{2} \\ 1 \end{pmatrix}$ ~~staat ⊥ e_0' en ⊥ e~~

Staat $\perp \vec{v}$ en $\perp \vec{w}$.

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Laat zien dat $W = \{ f(x) = a x^{10} + b x^{11} + c x^{12} \mid a, b, c \in \mathbb{C} \}$ een deelvectorruimte van $L^2([-1,1])$ is.

Als $f(x) = a x^{10} + b x^{11} + c x^{12}$, dan is
$(\lambda f)(x) = \lambda(a x^{10} + b x^{11} + c x^{12}) = (\lambda a) x^{10} + (\lambda b) x^{11} + (\lambda c) x^{12}$.
Dus $(\lambda f)(x) = a'' x^{10} + b'' x^{11} + c'' x^{12}$ met
$a'' = \lambda a$, $b'' = \lambda b$, $c'' = \lambda c$. Dus λf is in W.
Is ook $g(x) = a' x^{10} + b' x^{11} + c' x^{12}$ in W, dan is ook
$f + g\,(x) = (a + a') x^{10} + (b + b') x^{11} + (c + c') x^{12}$
een element van W.

Geef een orthogonale basis van W, met betrekking tot het gebruikelijke inproduct
$$\langle f, g \rangle = \int_{-1}^{1} \bar{f}(x) g(x)\, dx.$$

$b_0 = x^{10}$, $b_1 = x^{11}$, $b_2 = x^{12}$ is een basis, maar niet orthogonaal:

- $\langle b_0, b_1 \rangle = 0$ en $\langle b_1, b_2 \rangle = 0$ (éénintegraal oneven functie)

  maar $\langle b_0, b_2 \rangle = \int_{-1}^{1} x^{22}\, dx = \left[\frac{1}{23} x^{23} \right]_{-1}^{1} = \frac{2}{23}$

verder is $\|b_0\|^2 = \int x^{20}\, dx = \frac{2}{21}$

$\|b_1\|^2 = \int x^{22}\, dx = \frac{2}{23}$

$\|b_2\|^2 = \int x^{24}\, dx = \frac{2}{25}$

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Orthogonaliseren:

$$l_0' = l_0 = x^{10}$$

$$l_1' = l_1 - \frac{\langle l_0' l_1 \rangle}{\langle l_0' l_0' \rangle} l_0' = l_1 = x^{11}$$

$$l_2' = l_2 - \frac{\langle l_0' l_2 \rangle}{\langle l_0' l_0' \rangle} l_0' - \underbrace{\frac{\langle l_1' l_2 \rangle}{\langle l_1' l_1' \rangle}}_{=0} l_1' = l_2 - \frac{2/23}{2/21} l_0$$

$$= x^{12} - \frac{21}{23} x^{10}$$

$$l_0' = x^{10}, \quad l_1' = x^{11}, \quad l_2' = x^{12} - \frac{21}{23} x^{10}$$

Bereken de projectie van $f(x) = x$ op W.

$$P = \frac{\langle l_0' f \rangle}{\langle l_0' l_0' \rangle} \cdot l_0' + \frac{\langle l_1' f \rangle}{\langle l_1' l_1' \rangle} l_1' + \frac{\langle l_2' f \rangle}{\langle l_2' l_2' \rangle} l_2'$$

Nu is $l_0' = x^{10}$ even, dus $\int_{-1}^{1} \underbrace{x^{10} f(x)}_{\text{oneven}} dx = 0$

$l_2' = x^{12} - \frac{21}{23} x^{10}$ is ook even, dus $\langle l_2', f \rangle = 0$

Blijft over $\langle l_1' f \rangle = \int_{-1}^{1} x^{11} \cdot x \, dx = \left[\frac{1}{13} x^{13}\right]_{-1}^{1} = \frac{2}{13}$

en $\langle l_1', l_1' \rangle = \langle l_1, l_1 \rangle = 2/23$

Dus $P(f) = \frac{2/13}{2/23} l_1' = \frac{23}{13} x^{11}$

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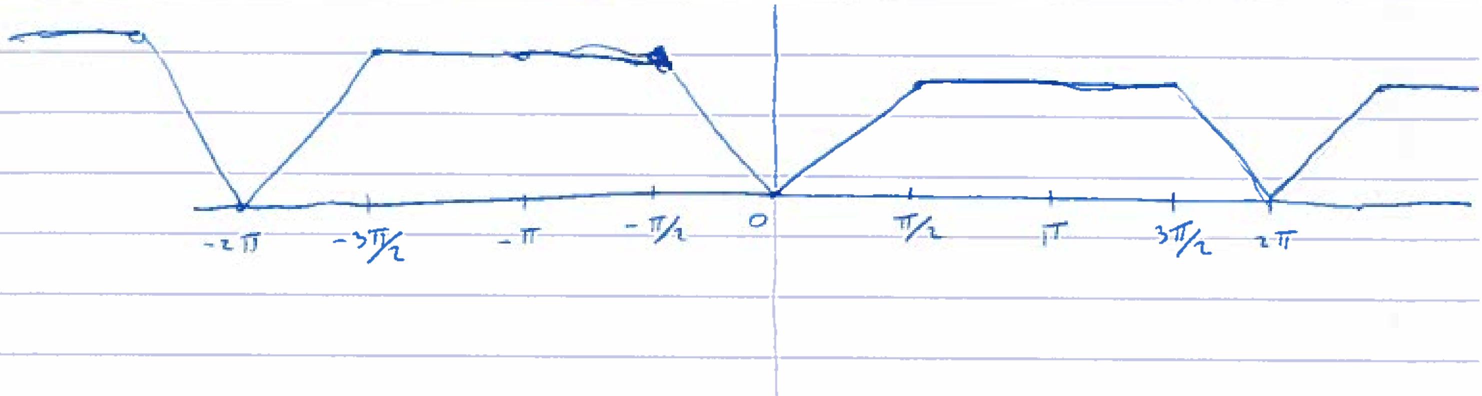
Opgave: Teken de grafiek van de 2π-periodieke functie met $f(x) = \begin{cases} |x| & \text{als } -\pi/2 \leq x \leq \pi/2 \\ \pi/2 & \text{als } \pi/2 \leq x \leq \pi \text{ en } -\pi \leq x \leq -\pi/2 \end{cases}$

(minstens 2 perioden



Bereken de coëfficiënten a_0, a_k, b_k waarvoor

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

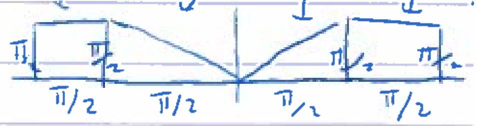
geldt.

- De b_k zijn 0 omdat $f(-x) = f(x)$ even is.

$k=0$

- $a_0 = \cancel{k}\, \frac{2}{2\pi} \cdot \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \cdot \left(\frac{\pi}{2} \cdot \frac{\pi}{2} + \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{\pi}{2} + \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{\pi}{2} + \frac{\pi}{2} \cdot \frac{\pi}{2} \right.$

  $a_0 = 3/4\,\pi$



$k \neq 0$

- $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\,dx =$ (gebruik dat f even is!)

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx)\,dx$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/2} x \cos(kx)\,dx + \int_{\pi/2}^{\pi} \cos(kx)\,dx \right)$$

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Nu is $$\int_{\pi/2}^{\pi} \cos(kx)\,dx = \left[\frac{1}{k}\sin(kx)\right]_{\pi/2}^{\pi} = \underbrace{\frac{\sin(k\pi)}{k}}_{=0} - \frac{\sin(k\pi/2)}{k}$$

$$= -\frac{\sin(k\pi/2)}{k}$$

en $$\int_0^{\pi/2} x\cos(kx)\,dx = \int_0^{\pi/2} x\, d\left(\frac{1}{k}\sin(kx)\right)$$

$$= \left[\frac{x}{k}\sin(kx)\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{k}\sin(kx)\,dx$$

$$= \frac{\pi}{2k}\sin(k\pi/2) + \left[\frac{1}{k^2}\cos(kx)\right]_0^{\pi/2}$$

$$= \frac{\pi}{2k}\sin(k\pi/2) + \frac{1}{k^2}\left(\cos(k\pi/2) - 1\right)$$

Dus als $k \neq 0$:

$$a_k = \sin(k\pi/2)\cdot\left(\frac{1}{k} - \frac{2}{\pi k}\right) + \frac{2}{\pi k^2}\left(\cos(k\pi/2) - 1\right)$$

Geef a_0, a_1, a_2, a_3 expliciet: $\sin(k\pi/2) = 0, 1, 0, -1$, kpn; $\cos(k\pi/2) = 1, 0, -1, 0$

$\cancel{a_0 = \frac{3}{4}\pi,\ a_1 = -\frac{2}{k},\ a_2 = \frac{-2}{\pi 2^2} = \frac{-1}{2\pi},}$

$\cancel{a_3 =}$

$a_0 = \frac{3}{4}\pi$, $a_1 = \left(1 - \frac{2}{\pi}\right) - \frac{2}{\pi} = 1 - \frac{4}{\pi}$

$a_2 = \frac{-2\cdot 2}{\pi\cdot 2^2} = -\frac{1}{\pi}$, $a_3 = -\left(\frac{1}{3} - \frac{2}{3\pi}\right) - \frac{2}{\pi\cdot 3^2}$

$$= -\frac{1}{3} + \frac{2}{3\pi} - \frac{2}{9\pi}$$

$$= -\frac{1}{3} + \frac{4}{9\pi}.$$

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Het Fourier Polynoom van graad n is gegeven

door $\quad P_2(x) = \frac{1}{2}a_0 + a_1\cos(x) + a_2\cos(2x)$,

met $\quad a_0 = \frac{3}{4}\pi$, $\quad a_1 = \sin(\pi/2)\left(1-\frac{2}{\pi}\right)\cdot \cancel{1} + \frac{2}{\pi}(1-1)$

$$= 1 - \frac{2}{\pi}$$

en $\quad a_2 = 0\cdot\left(\frac{1}{2} - \frac{2}{2\pi}\right) + \frac{2}{\pi\cdot 4}(-1-1) = -\frac{1}{\pi}$

Dus: $\quad P_2(x) = \frac{3}{8}\pi + \left(1-\frac{2}{\pi}\right)\underbrace{\cos(x)}_{=\frac{1}{2}(e^{ix}+e^{-ix})} \cancel{+} - \frac{1}{\pi}\underbrace{\cos(2x)}_{=\frac{1}{2}(e^{2ix}+e^{-2ix})}$

Vul in:

$$P_2(x) = - \frac{1}{2\pi}e^{-2ix} + \left(\frac{1}{2} - \frac{1}{\pi}\right)e^{-ix} + \frac{3}{8}\pi + \left(\frac{1}{2}-\frac{1}{\pi}\right)e^{+ix} - \frac{1}{2\pi}e^{2ix}$$

Dus $\quad c_{-2} = -\frac{1}{2\pi}$, $\quad c_{-1} = \frac{1}{2} - \frac{1}{\pi}$, $\quad c_0 = \frac{3}{8}\pi$, $\quad c_1 = \left(\frac{1}{2} - \frac{1}{\pi}\right)$,

$c_2 = -\frac{1}{2\pi}$.

zg fel

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De Fouriercoëfficiënten van $f(x) = \pi^2 - x^2$ zijn

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\, e^{-ikx}\, dx$$

$$= \left(\frac{\pi}{2}\int_{-\pi}^{\pi} e^{ikx}\, dx\right) + -\frac{1}{2\pi}\left(\int_{-\pi}^{\pi} x^2 e^{-ikx}\, dx\right)$$

$$\hookrightarrow = \frac{\pi}{2}\left[\frac{1}{ik} e^{ikx}\right]_{-\pi}^{\pi} = 0$$

als $k \neq 0$

$$= \frac{\pi}{2}\int_{-\pi}^{\pi} 1\, dx = \pi^2 \text{ als } k = 0$$

We berekenen $\int_{-\pi}^{\pi} x^2 e^{-ikx}\, dx = \int_{-\pi}^{\pi} x^2\, d\left(\frac{1}{-ik} e^{-ikx}\right)$

$$= \left[x^2 \cdot \frac{1}{-ik} e^{-ikx}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{-ik} \cdot 2x\, e^{-ikx}\, dx$$

als $k \neq 0$

$$= 0 + \frac{2}{ik}\int_{-\pi}^{\pi} x\, e^{-ikx}\, dx$$

$$= \frac{2}{ik}\int_{-\pi}^{\pi} x\, d\left(-\frac{1}{ik} e^{-ikx}\right)$$

$$= \frac{2}{ik}\left[-\frac{1}{ik} x\, e^{-ikx}\right]_{-\pi}^{\pi} + \frac{2}{ik}\int_{-\pi}^{\pi} e^{-ikx}\, dx$$

$$= \frac{2}{ik}\left(\frac{-\pi \cdot (-1)^k}{ik} - \frac{-\pi}{ik}(-1)^k\right) + \frac{2}{k^2}\underbrace{\left[e^{-ikx}\right]_{-\pi}^{\pi}}_{=0}$$

$$= \frac{4\pi}{k^2}(-1)^k \qquad \text{als } k \neq 0$$

als $k = 0$: $\int_{-\pi}^{\pi} x^2 \cdot 1\, dx = \left[\frac{1}{3} x^3\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^3$

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Dus: $c_0 = -\frac{1}{2\pi}\left(\cdot \frac{2}{3}\pi^3\right) + \pi^2 = \frac{2}{3}\pi^2$

$$c_k = -\frac{1}{2\pi} \cdot \frac{4\pi}{k^2}(-1)^k = \frac{2\cdot(-1)^{k+1}}{k^2}$$

Gebruik $\pi^2 - x^2 = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ voor $x = 0$

om $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ uit te rekenen.

$$\pi^2 - 0 = \lim_{N\to\infty}\left(\frac{2}{3}\pi^2 + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{2\cdot(-1)^{k+1}}{k^2}\cdot 1\right)$$

Dus $\frac{1}{3}\pi^2 = \lim_{N\to\infty}\left(-2\sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{(-1)^k}{k^2}\right) = -4 \lim_{N\to\infty} \sum_{k=1}^{N} \frac{(-1)^k}{k^2}$

Dus $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$