Werkcollege Mat Methoden

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Inleveropgave
do 31 maart

Opgave:

a) $\frac{d}{dt}\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix}$,

dus $\frac{d}{dt}\vec{v}_t = -i A \cdot \vec{v}_t$ voor $\vec{v}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$

en $-iA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

$H(\vec{v}) = A \cdot \vec{v}$ is hermitisch omdat $a_{ij} = -\bar{a}_{ji}$,
$\bar{0} = 0$, $\bar{i} = -i$.

b) Eigenwaarden van $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$:

$\det\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 + 1 = 0$ $\quad \lambda = \pm i$

Bij $\lambda = i$: $\quad \vec{v}_i = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Bij $\lambda = -i$: $\quad \vec{v}_{-i} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_+\begin{pmatrix} 1 \\ i \end{pmatrix} + c_-\begin{pmatrix} 1 \\ -i \end{pmatrix}$ geeft

$\left.\begin{aligned} c_+ + c_- &= x_0 \\ i(c_+ - c_-) &= y_0 \end{aligned}\right\}$ $\quad \begin{aligned} c_+ &= \tfrac{1}{2}(x_0 - iy_0) \\ c_- &= \tfrac{1}{2}(x_0 + iy_0) \end{aligned}$

Dus $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{1}{2}(x_0 - iy_0)\begin{pmatrix} 1 \\ i \end{pmatrix} + \left(\frac{1}{2}\right)(x_0 + iy_0)\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\Downarrow$

$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \frac{1}{2}(x_0 - iy_0)e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2}(x_0 + iy_0)e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$= \begin{pmatrix} x_0 \cos t + y_0 \sin(t) \\ -x_0 \sin t \quad y_0 \cos(t) \end{pmatrix}$

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c) $x_t^2 + y_t^2 = \left(x_0 \cos(t) + y_0 \sin(t)\right)^2 + \left(y_0 \cos(t) - x_0 \sin(t)\right)^2$

$= x_0^2 \cos^2 t + 2x_0 y_0 \cos t \sin t + y_0^2 \sin^2 t$
$+ y_0^2 \cos^2 t - 2 x_0 y_0 \cos t \sin t + x_0^2 \sin^2 t$

$= x_0^2 (\cos^2 t + \sin^2 t) + y_0^2 (\cos^2 t + \sin^2 t)$

$= x_0^2 + y_0^2$

Opgave: als (x_1, x_2, x_3, x_4) voldoet aan

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & -0{,}42 & -1{,}32 & -7{,}62 \\ 0{,}42 & 0 & -4{,}21 & -4{,}22 \\ 1{,}32 & 4{,}21 & 0 & -3{,}17 \\ 7{,}62 & 4{,}22 & 3{,}17 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

en $x_1^2 + \dots + x_4^2 = R^2$ op $t = 0$, wat is dit dan op tijdstip t?

Antw: Geen zinnig mens gaat hieraan rekenen. (of in ieder geval geen lui mens)

De vergelijking is van de vorm $\frac{d}{dt} V_t = -i A \cdot \vec{v}_t$

met $A = \begin{pmatrix} 0 & -0{,}42i & -1{,}52i & \dots \\ 0{,}42i & 0 & -4{,}21i & \dots \\ 1{,}32i & 4{,}21i & \dots & \\ \vdots & \vdots & & \end{pmatrix}$

A is hermitisch, $a_{ji} = \bar{a}_{ij}$.

Dus $\langle \vec{v}_t, \vec{v}_t \rangle = \langle \vec{v}_0, \vec{v}_0 \rangle$.

Maar $x_1, \dots, x_4$ is reëel, dus

$\langle V_{t=0}, V_{t=0} \rangle = |x_1(0)|^2 + \dots + |x_4(0)|^2 = x_1^2(0) + \dots + x_4^2(0)$
$= R^2$

Omdat $\frac{d}{dt} x_i$ reëel is voor alle t is ook
$x_i(t)$ reëel, en ~~$x_i(t)$~~
$\langle \vec{v}_t, \vec{v}_t \rangle = |x_1(t)|^2 + \dots + |x_4(t)|^2 = x_1^2(t) + \dots + x_4^2(t)$.

Omdat $\langle v(t=0), \vec{v}(t=0) \rangle = \langle \vec{v}(t), \vec{v}(t) \rangle$,
is dus $x_1^2(t) + \dots + x_4^2(t) = R^2$.

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Opg: Voor een vast getal c_g bekijken we

$$\frac{d}{dt}\Psi_t = -c_g \frac{d}{d\phi}\Psi_t$$

a) $\frac{d}{dt}\Psi_t = L\Psi_t$ met $L = -c_g \frac{d}{d\phi}$

$L\Psi_k = -c_g \frac{d}{d\phi} e^{ik\phi} = -ic_g k\, e^{ik\phi}$

dus $\lambda_k = -i c_g k$ is de eigenwaarde, $\omega_k = c_g k$

c) $\Psi_k(\phi,t) = e^{\lambda t}\Psi_k(\phi) = e^{-ic_g k t}\cdot e^{ik\phi}$

d) $\Psi_0 = \lim_{N\to\infty} \sum_{k=-N}^{N} c_k e^{ik\phi}$.

Dan $\Psi_t(\phi) = \lim_{N\to\infty} c_k e^{-ic_g k t} e^{ik\phi}$,

dus $c_k(t) = c_k e^{-ic_g k t}$.

e) $\Psi_t(\phi) = \lim_{N\to\infty} \sum_{k=-N}^{N} c_k \cdot e^{-ic_g k t}\cdot e^{ik\phi}$

$= \lim_{N\to\infty} \sum_{k=-N}^{N} c_k e^{ik(\phi - c_g t)}$

$= \Psi_0(\phi - c_g t)$

b) $v = \omega_k / k = c_g$

i) $H = -c_g i \frac{d}{d\phi}$ is een reëel veelvoud van de momentumoperator $P = -i\hbar \frac{d}{d\phi}$.

Inderdaad is $\|\Psi_t\|^2 = \int_{-\pi}^{\pi} |\Psi(\phi - c_g t)|^2 d\phi = \int_{-\pi}^{\pi} |\Psi(\phi)|^2 d\phi = \|\Psi_0\|^2$.

Opgave: Laat zien dat $H = -\frac{\hbar^2}{2m}(\partial_x^2 + \partial_y^2 + \partial_z^2) + V$ hermitisch is.

We laten zien dat $p_x = -i\hbar\,\partial_x$ hermitisch is.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{(-i\hbar\,\partial_x \psi_1)}\,\psi_2 \, dx\,dy\,dz =$$

$$i\hbar \cdot \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \left(\int_{-\infty}^{\infty} \partial_x \overline{\psi_1} \cdot \psi_2 \, dx \right)$$

$$\hookrightarrow = \left(\left[\overline{\psi_1}\,\psi_2\right]_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} \overline{\psi_1}\, \partial_x \psi_2(x,y,z)\, dx \right)$$

$$\hookrightarrow = 0$$

$$= -i\hbar \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \left(\overline{\psi_1}\, \partial_x \psi_2\right) dx =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\psi_1}(x,y,z) \cdot -i\hbar\,\partial_x \psi_2(x,y,z)\, dx\,dy\,dz.$$

Dus $\langle p_x \psi_1, \psi_2 \rangle = \langle \psi_1, p_x \psi_2 \rangle$.

Net zo zijn p_y en p_z hermitisch.

Dus p_x^2, p_y^2, p_z^2 zijn hermitisch.

Dus het reële veelvoud van hun som, $\frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$ is hermitisch.

Omdat V reëel is, is $\langle \psi_1, V\psi_2 \rangle = \iiint \overline{\psi_1}\, V \psi_2 \, dx\,dy\,dz$
$= \iiint \overline{V\psi_1}\,\psi_2 \, dx\,dy\,dz = \langle V\psi_1, \psi_2 \rangle$.

Dus V is ~~ook~~ hermitisch, en $H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V$ is hermitisch.