WC Do 24 mrt

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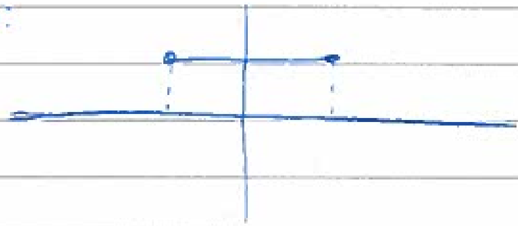
Opgave: $f(x) = \begin{cases} 1/2 & \text{als } |x| \le 1 \\ 0 & \text{als } |x| > 1 \end{cases}$, $f : \mathbb{R} \to \mathbb{C}$

a) Schets f

Antw:



b) Bereken $\mathcal{F}f(\omega)$

Antw: $\mathcal{F}f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \frac{1}{2} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \left[\frac{1}{-i\omega} e^{-i\omega x} \right]_{-1}^{1} \qquad \text{als } \omega \neq 0$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{2i\omega} \left(e^{-i\omega} - e^{i\omega} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\omega} \frac{e^{i\omega} - e^{-i\omega}}{2i}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\sin(\omega)}{\omega}$$

Als $\omega = 0$: $\mathcal{F}f(0) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \frac{1}{2}\, dx = \frac{1}{\sqrt{2\pi}}$

~~c) Is $\mathcal{F}f$ continu in $\omega = 0$?~~

~~A: Ja: $\lim_{\omega \to 0} \frac{\sin(\omega)}{\omega} = 1$, dus $\lim_{\omega \to 0} \mathcal{F}f(\omega) = \frac{1}{\sqrt{2\pi}}$.~~

c) Geef de ontbinding van $f(x)$ in golven $e^{i\omega x}$ voor $x \neq \pm 1$

A: Dan geldt $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{i\omega x} \, d\omega$

dus

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(\omega)}{\omega} e^{i\omega x} \, d\omega$$

$\downarrow$

$= 1/2$ als $|x| < 1$

$= 0$ als $|x| > 1$

d) Wat gebeurt er als $x = \pm 1$?

A: Dan is $\frac{1}{2}\left(f(x_+) + f(x_-)\right) = \frac{1}{2}\left(1/2 + 0\right) = 1/4$

Dus: $\frac{1}{4} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(\omega)}{\omega} e^{i\omega} \, d\omega$ voor $x = +1$

$\frac{1}{4} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(\omega)}{\omega} e^{-i\omega} \, d\omega$ voor $x = -1$

Opgave ~~A~~ Is $\mathcal{F}f$ continu in $\omega = 0$?

A: Ja. $\mathcal{F}f(\omega = 0) = \frac{1}{\sqrt{2\pi}}$

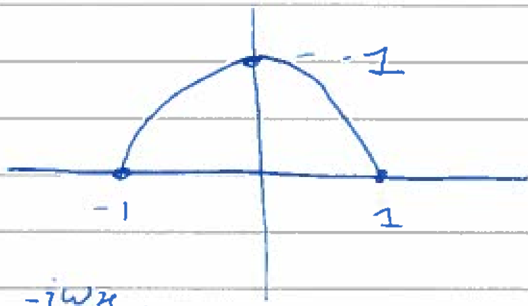
$$\lim_{\omega \to 0} \mathcal{F}f(\omega) = \lim_{\omega \to 0} \frac{\sin(\omega)}{\omega \cdot \sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}$$

Is $\lim_{\omega \to \pm\infty} \mathcal{F}f(\omega) = 0$?

Ja: $\left|\frac{\sin(\omega)}{\omega}\right| < \frac{1}{|\omega|}$, en $\lim_{\omega \to \pm\infty} \frac{1}{|\omega|} = 0$.

Opgave: $f(x) = \begin{cases} 1-x^2 & \text{als } |x| \le 1 \\ 0 & \text{als } |x| > 1 \end{cases}$

a) Schets f



b) Bereken $\mathcal{F}f(\omega)$

Antw: $\mathcal{F}f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{-i\omega x}\, dx$$

Nu is $\int_{-1}^{1} e^{-i\omega x}\, dx = \left[\frac{1}{-i\omega} e^{-i\omega x}\right]_{-1}^{1} = \frac{e^{-i\omega} - e^{i\omega}}{-i\omega} = 2\,\frac{\sin(\omega)}{\omega}$.

Voor de integraal $\int_{-1}^{1} x^2 e^{-i\omega x}\, dx$ gebruiken we partiele integratie, twee maal: als $\omega \neq 0$

$$\int_{-1}^{1} x^2 e^{-i\omega x}\, dx = \int_{-1}^{1} x^2\, d\left(-\frac{1}{i\omega} e^{-i\omega x}\right) =$$

$$\left[-\frac{x^2}{i\omega} e^{-i\omega x}\right]_{-1}^{1} + \int_{-1}^{1} \frac{1}{i\omega} e^{-i\omega x}\, \cancel{dx} \cdot 2x\, dx =$$

$$\left(\frac{-e^{-i\omega}}{i\omega} + \frac{e^{i\omega}}{i\omega}\right) + \frac{2}{\cancel{i}\omega^2} \int_{-1}^{1} x\, d\left(e^{-i\omega x}\right) =$$

$$= 2\,\frac{\sin\omega}{\omega} + \frac{2}{\cancel{i}\omega^2}\left(\left[x\, e^{-i\omega x}\right]_{-1}^{1} - \int_{-1}^{1} e^{-i\omega x}\, dx\right)$$

$$= 2\,\frac{\sin\omega}{\omega} + \frac{2}{\cancel{i}\omega^2}\left(e^{-i\omega} + e^{i\omega} - \left[\frac{1}{-i\omega} e^{-i\omega x}\right]_{-1}^{1}\right)$$

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$$= 2\frac{\sin w}{w} + \frac{4\cos(w)}{w^2} - \frac{4\sin(w)}{w^3}$$

Dus $\mathcal{F}f_w = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} 1\, e^{-iwx}\,dx - \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} x^2 e^{-iwx}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\left(2\frac{\sin w}{w} - 2\frac{\sin w}{w} - 4\frac{\cos w}{w^2} - \frac{4\sin w}{w^3}\right)$$

$$= \frac{-4}{\sqrt{2\pi}}\left(\frac{\cos w}{w^2} - \frac{\sin w}{w^3}\right)$$

$$= 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin w}{w^3} - \frac{\cos w}{w^2}\right) \quad \text{voor } w \neq 0$$

$$\mathcal{F}f\,(w=0) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} 1 - x^2\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{1}{\sqrt{2\pi}}\cdot\frac{4}{3} = \frac{2}{3}\sqrt{\frac{2}{\pi}}$$

c) Is $\mathcal{F}f$ continu in $w = 0$?

A: Ja: $\mathcal{F}f\,(w=0) = \frac{2}{3}\cdot\sqrt{\frac{2}{\pi}}$

en $\lim_{w\to 0}\mathcal{F}f(w) = 2\sqrt{\frac{2}{\pi}}\left(\lim_{w\to 0}\left(\frac{w - \frac{1}{6}w^3 + \cdots}{w^3} - \frac{1 - \frac{1}{2}w^2 + \cdots}{w^2}\right)\right)$

$$= 2\sqrt{\frac{2}{\pi}}\left(-\frac{1}{6} + \frac{1}{2}\right)$$

$$= 2\sqrt{\frac{2}{\pi}}\cdot\frac{1}{3} \qquad \&$$

c) $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{F}f(w)\,e^{+iwx}\,dw = \int_{-\infty}^{\infty}\frac{2}{\pi}\left(\frac{\sin w}{w^3} - \frac{\cos w}{w^2}\right)e^{iwx}\,dw$

Opgave: $\mathcal{F}f(\omega) = \begin{cases} \frac{2}{3}\sqrt{\frac{2}{\pi}} & \text{als } \omega = 0 \\ 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega}{\omega^3} - \frac{\cos\omega}{\omega^2}\right) & \text{als } \omega \neq 0 \end{cases}$

laat zien dat $\lim_{\omega\to\pm\infty} \mathcal{F}f(\omega) = 0$, en dat $\mathcal{F}f(\omega)$ continu is
(in 0, voor $\omega \neq 0$ is dit wel duidelijk.)

Omdat $\left|\frac{\sin\omega}{\omega^3}\right| < \frac{1}{|\omega|^3}$ en $\left|\frac{\cos\omega}{\omega^2}\right| < \frac{1}{|\omega|^2}$,

is $\left|\lim_{\omega\to\pm\infty} \mathcal{F}f(\omega)\right| < \lim_{\omega\to\pm\infty} \frac{1}{|\omega|^3} + \frac{1}{|\omega|^2} = 0$.

Dus $\lim_{\omega\to\pm\infty} \mathcal{F}f(\omega) = 0$.

Voor $\omega = 0$ hebben we

$$\lim_{\omega\to 0} \mathcal{F}f(\omega) = \lim_{\omega\to 0} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega}{\omega^3} - \frac{\cos\omega}{\omega^2}\right)$$

$$= \lim_{\omega\to 0} 2\sqrt{\frac{2}{\pi}}\left(-\frac{1-\frac{1}{2}\omega^2+\frac{1}{24}\omega^4\ldots}{\omega^2} + \frac{\omega-\frac{1}{6}\omega^3+\frac{1}{120}\omega^5+\ldots}{\omega^3}\right)$$

$$= \lim_{\omega\to 0} 2\sqrt{\frac{2}{\pi}}\left(\frac{1}{2} - \frac{1}{24}\omega^2\ldots + \left(-\frac{1}{6} + \frac{1}{120}\omega^2 + \ldots\right)\right)$$

$$= 2\sqrt{\frac{2}{\pi}}\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{2}{3}\sqrt{\frac{2}{\pi}}.$$

Dit is precies $\mathcal{F}f(\omega=0)$, dus ze is continu.