Opgave:

a) Wat is de Fouriergetransformeerde van $f(x) = e^{-x^2/2}$?

Antw: $\mathcal{F}f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-i\omega x} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 - i\omega x} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\omega)^2 - \frac{1}{2}\omega^2} dx$

$= e^{-\frac{1}{2}\omega^2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\omega)^2} dx$

$= e^{-\frac{1}{2}\omega^2} \cdot \frac{1}{\sqrt{2\pi}} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du}_{\sqrt{2\pi}}$ $(u = x + i\omega)$

$= e^{-\frac{1}{2}\omega^2}$

b) Wat is de Fouriergetransformeerde van $f_\mu(x) = e^{-(x-\mu)^2/2}$?

Antw: • Dit is een verschuiving over μ;

dus $\mathcal{F}f_\mu(\omega) = e^{-i\mu\omega} \mathcal{F}f(\omega) = e^{-\frac{1}{2}\omega^2 - i\omega\mu}$

• Of direct: $\mathcal{F}f_\mu(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\mu)^2} e^{-i\omega x} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\mu+i\omega)^2 - i\omega\mu - \frac{1}{2}\omega^2} dx$

$= e^{-\frac{1}{2}\omega^2 - i\mu\omega} \cdot \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\mu+i\omega)^2} dx}_{= \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du = \sqrt{2\pi}}$

$= e^{-\frac{1}{2}\omega^2 - i\mu\omega}$

met $u = x - \mu + i\omega$

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c) Wat is de Fourier getransformeerde van

$$f_{\mu,\sigma}(x) = e^{-\frac{1}{2}(x-\mu)^2/2\sigma^2}$$

A: $$\mathcal{F} f_{\mu,\sigma}(\omega) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\mu)^2/2\sigma^2} e^{-i\omega x} \, dx \qquad \tilde{x} = \frac{x-\mu}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\tilde{x}^2} e^{-i\omega\sigma\tilde{x}} e^{-i\omega\mu} \cdot (\sigma \, d\tilde{x}) \qquad x = \sigma\tilde{x} + \mu$$

$$= \sigma e^{-i\omega\mu} \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\tilde{x}^2} e^{-i(\omega\sigma)\tilde{x}} \, d\tilde{x}$$

$$= \sigma e^{-i\omega\mu} \cdot \mathcal{F} f_{0,1}(\omega\sigma)$$

$$= \sigma e^{-i\omega\mu} e^{-\frac{1}{2}(\sigma\omega)^2}$$

d) Wat is dus de Fourier getransformeerde van de Gaussische kansverdeling $\tilde{f}_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$?

A: $\mathcal{F}$ is lineair, dus $\mathcal{F}(\tilde{f}_{\mu,\sigma}) = \frac{1}{\sigma\sqrt{2\pi}} \mathcal{F} f_{\mu,\sigma}$

$$\mathcal{F} \tilde{f}_{\mu,\sigma} = \frac{1}{\sqrt{2\pi}} e^{-i\omega\mu} e^{-\frac{1}{2}(\sigma\omega)^2}$$

Opgave: Als f absoluut integreerbaar is, dan is de Fourier getransformeerde van $f_\sigma(x) = f(x/\sigma)$ gelijk aan $\mathcal{F}f_\sigma(\omega) = \sigma \mathcal{F}f(\sigma\omega)$

Bewijs: $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x/\sigma)\, e^{-i\omega x}\, dx = \quad \left(\begin{array}{l} u = x/\sigma \\ dx = \sigma\, du \end{array} \right.$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega\sigma u} \cdot \sigma\, du =$$

$$\sigma \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i(\omega\sigma)\cdot u}\, du$$

$$= \sigma \mathcal{F}f(\sigma\omega)$$

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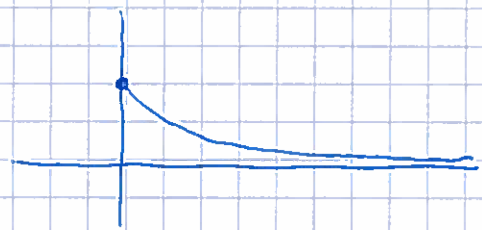
Opgave : $f(x) = \begin{cases} e^{-ax} & x \geq 0 \\ 0 & x < 0 \end{cases}$

a)



b) $\mathcal{F}f(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(ax+iwx)} dx$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{a+iw} e^{-(a+iw)x} \right]_0^{\infty} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a+iw}$

c) $a(w) = \frac{1}{2}\left(\mathcal{F}f(w) + \mathcal{F}f(-w)\right) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a+iw} + \frac{1}{a-iw} \right)/2$

$= \frac{1}{\sqrt{2\pi}} \left(\frac{a}{a^2+w^2} \right)$

$b(w) = \frac{i}{2} \left(\mathcal{F}f(w) - \mathcal{F}f(-w) \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{i}{a+iw} - \frac{i}{a-iw} \right) \cdot \frac{1}{2}$

$= \frac{1}{\sqrt{2\pi}} \frac{w}{a^2+w^2}$

Dus $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} a(w) \cos(wx) dw + \sqrt{\frac{2}{\pi}} \int_0^{\infty} b(w) \sin(wx) dw$

geeft $e^{-ax} = \frac{1}{\pi} \int_0^{\infty} \frac{a \cos wx}{a^2+w^2} dw + \frac{1}{\pi} \int_0^{\infty} \frac{w \sin(wx)}{a^2+w^2} dw$

en $f(-x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} a(w) \cos(wx) dw$

$- \sqrt{\frac{2}{\pi}} \int_0^{\infty} b(w) \sin(wx) dw = 0,$

dus beide integralen zijn hetzelfde!

Dus $\int_0^{\infty} \frac{w \sin wx}{a^2+w^2} dw = \frac{\pi}{2} e^{-ax}$, $\int_0^{\infty} \frac{\cos(wx)}{a^2+w^2} dw = \frac{\pi}{2a} e^{-ax}$.