WC 7 afl.

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Opgave:

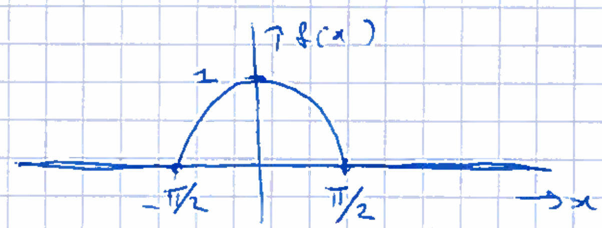
De functie $f:\mathbb{R}\to\mathbb{C}$ wordt gegeven door

$$f(x)=\begin{cases}\cos(x) & \text{als } -\pi/2\leq x\leq \pi/2\\ 0 & \text{als } |x|>\pi/2 .\end{cases}$$

De functie $g:\mathbb{R}\to\mathbb{C}$ wordt gegeven door

$$g(x)=\frac{\cos(\frac{\pi}{2}x)}{x^2-1} \quad \text{als } x\neq\pm 1 .$$

a) schets de functies f. (meld of dan f en g geen van beide periodiek zijn.)



b) Bereken de Fourier getransformeerde $\mathcal{F}f(\omega)$ voor $\omega\neq\pm1$.

$$\mathcal{F}f(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2}\cos(x)\,e^{-i\omega x}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2}\frac{1}{2}\left(e^{ix}+e^{-ix}\right)e^{-i\omega x}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2}\frac{1}{2}e^{i(1-\omega)x}+\frac{1}{2}e^{-i(1+\omega)x}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{2}\cdot\frac{1}{i(1-\omega)}\left[e^{i(1-\omega)x}\right]_{-\pi/2}^{\pi/2}$$

$$+\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{2}\cdot\frac{-1}{i(1+\omega)}\left[e^{-i(1+\omega)x}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{i(1-w)} \left(i\, e^{-\frac{\pi}{2} i w} + i\, e^{i\frac{\pi}{2} w} \right)$$

$$+ \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{i(1+w)} \left(i\, e^{-\frac{\pi}{2} i w} + i\, e^{i\frac{\pi}{2} w} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cos\left(\frac{\pi}{2} w\right) \left(\frac{1}{1-w} + \frac{1}{1+w} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2\cos\left(\frac{\pi}{2} w\right)}{1-w^2}$$

c) Hoe luidt de Fourier inverseformule voor g?

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}f(w)\, e^{iwx}\, dw$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2} w\right)}{1-w^2}\, e^{iwx}\, dw$$

d) Bereken $\mathcal{F}f(1)$ en $\mathcal{F}f(-1)$

$$\mathcal{F}f(1) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1 + e^{-2ix}\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\mathcal{F}f(-1) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(e^{2ix} + 1\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

e) Wat is de Fourier getransformeerde van g?
(Hint: kijk nog eens naar onderdeel b&c.)

A: $\mathcal{F}f(w) = -\frac{2}{\sqrt{2\pi}} \frac{\cos\left(\frac{\pi}{2} w\right)}{w^2-1} = -\frac{2}{\sqrt{2\pi}} g(w)$,

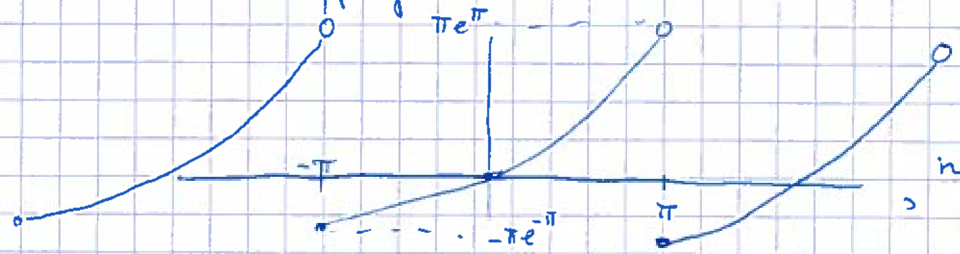
~~dus $\mathcal{F}^2 f(w)$~~ dus

$$\mathcal{F}g(y) = -\frac{\sqrt{2\pi}}{2} \mathcal{F}(\mathcal{F}f)(y)$$

$$= -\sqrt{\frac{\pi}{2}}\, f(-y) = \begin{cases} -\sqrt{\frac{\pi}{2}} \cos(y) & \text{als } |y| < \frac{\pi}{2} \\ 0 & \text{als } |y| > \frac{\pi}{2} \end{cases}$$

De 2π-periodieke functie $f : \mathbb{R} \to \mathbb{C}$ wordt gegeven door $f(x) = x e^{x}$ voor $-\pi \le x \le \pi$.

a) Schets de grafiek van f. Is f continu in $x = \pi$?



niet continu.

b) Voor welke Fourier coëfficiënten c_k is || geldt

$$f(x) = \lim_{N\to\infty} \sum_{k=-N}^{N} c_k e^{ikx}$$

~~En voor welke x geldt~~

Voor x $-\pi < x < \pi$ geldt dus $\lim_{N\to\infty} \sum_{k=-N}^{N} c_k e^{ikx} = x e^{x}$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{x} e^{-ikx}\, dx =$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} x\, e^{(1-ik)x}\, dx =$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} x\, d\left(\frac{e^{(1-ik)x}}{(1-ik)} \right) =$$

$$\frac{1}{2\pi} \cdot \left[\frac{x \cdot e^{(1-ik)x}}{1-ik} \right]_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{(1-ik)x}}{(1-ik)}\, dx$$

$$= \frac{1}{2\pi} \left[\frac{x e^{(1-ik)x}}{1-ik} - \frac{e^{(1-ik)x}}{(1-ik)^2} \right]_{-\pi}^{\pi}$$

$$= \cancel{\frac{1}{2\pi} \left(\frac{1}{1-ik} - \frac{1}{} \right.}$$

$$= \frac{1}{2\pi} \cdot \left(\frac{\pi e^{\pi} \cdot (-1)^k + \pi e^{-\pi} (-1)^k}{1-ik} - \frac{e^{\pi}(-1)^k - e^{-\pi}(-1)^k}{(1-ik)^2} \right)$$

$$= \frac{(-1)^k}{2} \left(e^{\pi} + e^{-\pi} \right) \cdot \frac{1}{1-ik} + \frac{(-1)^k}{2\pi} \left(e^{-\pi} - e^{\pi} \right) \cdot \frac{1}{(1-ik)^2}.$$

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c) Wat is $\lim_{N\to\infty} \sum_{k=-N}^{N} c_k(-1)^k$?

A: De functie f is niet continu in $x = \pi$,

dus de Fourierreeks $P_N(\pi) = \sum_{k=-N}^{N} c_k e^{ik\pi} = \sum_{k=-N}^{N} c_k(-1)^k$

convergeert naar

$$\frac{1}{2}\left(f(x_+)+f(x_-)\right) = \frac{1}{2}\left(-\pi e^{-\pi}+\pi e^{\pi}\right) = \frac{e^{\pi}-e^{-\pi}}{2}\cdot\pi .$$

En wat is $\lim_{N\to\infty} \sum_{k=-N}^{N} c_k$?

~~Hier is f continu, dus je~~
~~($\lim x = 0$~~

De functie $f(x)$ is gewoon continu in $x=0$,

dus $\lim_{N\to\infty} \sum_{k=-N}^{N} c_k \cdot \underset{x=0}{e^{ik\cdot x}} = \underset{x=0}{x e^{x}} = 0 .$

Opgave

$$\partial_t f_t(x) = \underbrace{(-\partial_x^3 + \partial_x)}_{L} f_t(x)$$

a) Wat is $\mathcal{F} f_t(x)$ in termen van $\mathcal{F} f_0(x)$?

A: Er geldt $f_t(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F} f_t(w) e^{iwx} dw$,

dus
$$\partial_t f_t(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\partial_t \mathcal{F} f_t(w)\right) e^{iwx} dw$$

$$L f_t(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F} f_t(w)\, L(e^{iwx})\, dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F} f_t(w) \underbrace{(iw^3 + iw)}_{\lambda(w)} e^{iwx} dw$$

Dus $\partial_t \mathcal{F} f_t(w) = i(w^3 + w) \mathcal{F} f_t(w)$,

los op met $\mathcal{F} f_t \big|_{t=0} (w) = \mathcal{F} f_0(w)$,

$$\mathcal{F} f_t(w) = \mathcal{F} f_0(w) e^{i(w^3+w)t}$$

b)
$$\int_{-\infty}^{\infty} |f_0(x)|^2 dx = \int_{-\infty}^{\infty} |\mathcal{F} f_0(w)|^2 dw$$

en

$$\int_{-\infty}^{\infty} |f_t(x)|^2 dx = \int_{-\infty}^{\infty} |\mathcal{F} f_t(w)|^2 dw$$

$$= \int_{-\infty}^{\infty} |\mathcal{F} f_0(w) \cdot e^{i(w^3+w)t}|^2 dw$$

$\hookrightarrow$ norm!

$$= \int_{-\infty}^{\infty} |\mathcal{F} f_0(w)|^2 \cdot \underbrace{|e^{i(w^3+w)t}|^2}_{=1} dw$$

$$= \int_{-\infty}^{\infty} |\mathcal{F} f_0(w)|^2 dw$$

$$= \int_{-\infty}^{\infty} |f_0(x)|^2 dx .$$

Opgave:

$$b_{k_x k_y} = e^{i(x k_x + y k_y)}$$

a) $$\langle b_{k_x k_y}, b_{k_x' k_y'} \rangle = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-i(x k_x + y k_y)} e^{i(x k_x' + y k_y')} \, dx\, dy$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i x(k_x' - k_x)} e^{i y(k_y' - k_y)} \, dx\, dy$$

$$= \int_{-\pi}^{\pi} e^{i x(k_x' - k_x)} \, dx \left(\int_{-\pi}^{\pi} e^{i(k_y' - k_y) y} \, dy \right)$$

↳ hangt niet van y af

$$= \left(\int_{-\pi}^{\pi} e^{i(k_x' - k_x) x} \, dx \right) \cdot \left(\int_{-\pi}^{\pi} e^{i(k_y' - k_y) y} \, dy \right)$$

$= 2\pi$ als $k_x' = k_x$ $\qquad$ $= 2\pi$ als $k_y' = k_y$

$= 0$ als $k_x' \neq k_x$ $\qquad$ $= 0$ als $k_y' \neq k_y$

want $\int_{-\pi}^{\pi} 1\, dx = 2\pi$ $\qquad$ Om dezelfde reden.

$$\int_{-\pi}^{\pi} e^{i(k_x' - k_x) x} \, dx =$$

$$\left[\frac{1}{i(k_x' - k_x)} e^{i(k_x' - k_x) x} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{i(k_x' - k_x)} \left((-1)^{k_x' - k_x} - (-1)^{k_x' - k_x} \right)$$

$$= 0$$

Dus $\langle b_{k_x k_y}, b_{k_x' k_y'} \rangle = 4\pi^2$ als $k_x = k_x'$ en $k_y = k_y'$

en $\langle b_{k_x k_y}, b_{k_x' k_y'} \rangle = 0$ als $k_x \neq k_x'$ of $k_y \neq k_y'$ (of allebei)

b) $c_{k_1,k_2} = \frac{\langle b_{k_1,k_2}, f \rangle}{\| b_{k_1,k_2} \|^2}$

$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \delta(x,y)\, e^{-ik_x x - ik_y y}\, dx\, dy$