WC 4 opgav

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Opgave: Stel $T_0(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x/\sigma)^2}$

a) $$\mathcal{F}T_0(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x/\sigma)^2} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x/\sigma + i\omega\sigma)^2 - \frac{1}{2}\omega^2\sigma^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot \Big(\underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\tilde{x}^2} \cdot \sigma\, d\tilde{x}}_{=\sigma\sqrt{2\pi}}\Big) \cdot e^{-\frac{1}{2}\sigma^2\omega^2}$$

(met $\tilde{x} = x/\sigma + i\omega\sigma$, $d\tilde{x} = \frac{1}{\sigma}dx$)

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2\omega^2}.$$

b) Dus $\mathcal{F}T_t(\omega) = \mathcal{F}T_0(\omega)\, e^{-\frac{\kappa}{c\rho}\omega^2 t}$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2\omega^2} e^{-\frac{\kappa}{c\rho}t\,\omega^2}$$

Eindelijk $\mathcal{F}T_t(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma_t^2\omega^2}$

met $\boxed{\sigma_t^2 = \sigma^2 + \frac{2\kappa}{c\rho}t}$

c) $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma_t^2\omega^2}$ is de Fouriergetransformeerde

van $\frac{1}{\sigma_t\sqrt{2\pi}} e^{-\frac{1}{2}(x/\sigma_t)^2}$,

dus $$T(t,x) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2 + \frac{2\kappa}{c\rho}t}} e^{-\frac{1}{2}\left(x^2/(\sigma^2 + \frac{2\kappa}{c\rho}t)\right)}$$

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Controleer dat dit een oplossing is:

$$\cdot\ \partial_t T_t(x) = \frac{1}{\sqrt{2\pi}} \cdot \left(-\tfrac{1}{2}\right)\left(\sigma^2 + \frac{2\kappa}{c\rho} t\right)^{-\frac{3}{2}} \cdot \left(\frac{2\kappa}{c\rho}\right) e^{-\frac{1}{2} x^2 / (\sigma^2 + \frac{2\kappa}{c\rho} t)}$$

$$+ \frac{1}{\sqrt{2\pi}} \cdot \left(\sigma^2 + \frac{2\kappa}{c\rho} t\right)^{-\frac{1}{2}} \cdot -\frac{x^2}{2} \cdot (-1) \cdot \left(\sigma^2 + \frac{2\kappa}{c\rho} t\right)^{-2} \cdot \frac{2\kappa}{c\rho} \times e^{-\frac{1}{2} x^2 / (\sigma^2 + \frac{2\kappa}{c\rho} t)}$$

$$= -\frac{1}{\sqrt{2\pi}} \cdot \frac{\kappa}{c\rho} \cdot \left(\sigma^2 + \frac{2\kappa}{c\rho} t\right)^{-3/2} e^{-\frac{1}{2} x^2 / (\sigma^2 + \frac{2\kappa}{c\rho} t)}$$

$$+ \frac{1}{\sqrt{2\pi}} \cdot \frac{\kappa}{c\rho} x^2 \left(\sigma^2 + \frac{2\kappa}{c\rho} t\right)^{-5/2} e^{-\frac{1}{2} x^2 / (\sigma^2 + \frac{2\kappa}{c\rho} t)}$$

$$\cdot\ \frac{\kappa}{c\rho} \partial_x^2 T_t(x) = \frac{\kappa}{c\rho} \partial_x \left(\frac{1}{\sqrt{2\pi}} \left(\sigma^2 + \frac{2\kappa}{c\rho} t\right)^{-1/2} \cdot \frac{-x}{(\sigma^2 + \frac{2\kappa}{c\rho} t)} e^{-\frac{1}{2} x^2 / (\sigma^2 + \frac{2\kappa}{c\rho} t)} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\kappa}{c\rho} \cdot \left(\sigma^2 + \frac{2\kappa}{c\rho} t\right)^{-3/2} \left(-1 + \left(\sigma^2 + \frac{2\kappa}{c\rho} t\right)^{-1}\right) e^{-\frac{1}{2} x^2 / (\sigma^2 + \frac{2\kappa}{c\rho} t)}.$$

Dus inderdaad is $\partial_t T(t,x) = \frac{\kappa}{c\rho} \partial_x^2 T_t(t,x)$

als $T(t,x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\sigma^2 + \frac{2\kappa}{c\rho} t}} e^{-\frac{1}{2} x^2 / (\sigma^2 + \frac{2\kappa}{c\rho} t)}$

Opgave: De Schrödinger vergelijking voor een vrij deeltje op een lijn is $i\hbar\,\partial_t \psi = -\frac{\hbar^2}{2m}\partial_x^2 \psi$

a) Als $\mathcal{F}\psi_0(\omega)$ de Fouriergetransformeerde van $\psi_0(x)$ is, wat is dan $\mathcal{F}\psi_t(\omega)$?

A: $$\psi_t(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}\psi_t(\omega)\, e^{i\omega x}\, d\omega,$$

dus $$i\hbar\,\partial_t \psi_t(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\hbar\,\partial_t \mathcal{F}\psi_t(\omega)\, e^{i\omega x}\, d\omega$$

$$-\frac{\hbar^2}{2m}\partial_x^2 \psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} +\frac{\hbar^2\omega^2}{2m}\mathcal{F}\psi_t(\omega)\, e^{i\omega x}\, d\omega$$

en $$i\hbar\,\partial_t \mathcal{F}\psi_t(\omega) = \frac{\hbar^2\omega^2}{2m}\mathcal{F}\psi_t(\omega),$$

$$\partial_t \mathcal{F}\psi_t(\omega) = -i\left(\frac{\hbar\omega^2}{2m}\right)\mathcal{F}\psi_t(\omega)$$

Dus $$\mathcal{F}\psi_t(\omega) = \mathcal{F}\psi_0(\omega)\cdot e^{-i\frac{\hbar\omega^2}{2m}t}$$

b) Stel dat $\psi_0(x) = \frac{1}{\sigma\sqrt{4\pi}}\, e^{-\frac{1}{2}(x/\sigma)^2}$.

Wat is dan $\psi_t(x)$?

Hint: Kijk goed naar opgave 5.4; de warmte vergelijking $\partial_t T_t = \frac{\kappa}{c\rho}\partial_x^2 T_t$ en

s.v. $\partial_t \psi_t = \frac{i\hbar}{2m}\partial_x^2 \psi_t$

zijn broertjes, met $\frac{\kappa}{c\rho} \leftrightarrow \frac{i\hbar}{2m}$

e) Oplossing:

$$\mathcal{F}\psi_0(\omega) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}\sigma^2\omega^2},$$

dus $$\mathcal{F}\psi_t(\omega) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}\sigma^2\omega^2} \cdot e^{-\frac{i\hbar}{2m}\omega^2 t},$$

dus $$\mathcal{F}\psi_t(\omega) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(\sigma^2 + \frac{i\hbar}{m} t)\omega^2}$$

$$= \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}\sigma_t^2\omega^2}$$

voor $\sigma_t^2 = \sigma^2 + \frac{i\hbar}{m} t$ (complex getal!)

Er geldt dus

$$\psi_t(\omega) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2} x^2/\sigma_t^2}$$

$$\cancel{= \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2 + \frac{i\hbar}{m} t}\right)}$$

$$= \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2\left(\frac{\sigma^2}{\sigma^4 + \frac{\hbar^2}{m^2}t^2} - i\frac{\hbar/m\, t}{\sigma^4 + \frac{\hbar^2}{m^2}t^2}\right)}$$

$$= \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2 \cdot \frac{\sigma^2}{\sigma^4 + \frac{\hbar^2}{m^2}t^2}} \cdot e^{\frac{1}{2} i x^2 \cdot \frac{\hbar/m\, t}{\sigma^4 + \frac{\hbar^2}{m^2}t^2}}.$$