Entanglement of symmetric Werner states

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Statement of the problem

We consider a quantum system consisting of n identical, but distinguishable subsystems ("particles") described by Hilbert spaces of dimension d.

A state on such a system is called a Werner state if it is invariant under the global unitary rotation of all the individual Hilbert spaces together. It is called symmetric if is invariant for permutation of the particles.

A state is called entangled if it can not be written as a convex combination of product states.

For a given symmetric Werner state, we want to find out if it is entangled or not.

And also, whether there is a relation between entangement and extendability.

Motivation

Entanglement is a central issue in quantum information theory.

The study of n party-entanglement is considered difficult. It is complicated by the fact that the state space of n systems of size d has a large dimension: $d^{2n} - 1$.

The number of parameters is greatly reduced by restricting attention to the symmetric Werner states. The dimension d drops out entirely, and the number of parameters becomes (one less than) the number of possible partitions of the n particles.

For example, for 2 quantum identical systems of arbitrary size d there is only one parameter.

An advantage of this restraint is that we can lean on a vast body of results from classical mathematics: the representation theory of S_n and $SU(d)$, as pioneered by Frobenius, Schur, Weyl, Littlewood, ...

But also some relatively recent work in multilinear algebra turns out to be relevant to our question.

Extendability relates to the 'monogamy' property of entanglement.

Overview of the talk

- Entanglement of Werner states for $n = 2$;
- Symmetric Werner states on n particles;
- Separable symmetric Werner states and immanants;
- The case $n = 3$;
- The 'shadow' of the product states and its behaviour for general n;
- Schur's inequality and Lieb's conjecture;
- The case $n = 4$;
- The case $n = 5$: Hope crashed.
- Extendability of symmetric Werner states
- A quantum de Finetti theorem of Christandl et al.

Symmetries in the two-particle case

$$\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d \ .$$

Symmetry group: $u \in SU(d)$ acting as $u \otimes u$.

This action commutes with the "Flip" operator:

$$F : \quad \varphi \otimes \psi \mapsto \psi \otimes \varphi \ .$$

Eigenspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of F are invariant for $SU(d)$:

$$\mathcal{H}_+ := \mathrm{span}\{\, \psi \otimes \psi \,|\, \psi \in \mathbb{C}^d \,\}$$

$$\text{basis:} \quad \{\, e_i \otimes e_i \,|\, 0 \le i \le d \,\} \cup \left\{ \frac{1}{\sqrt{2}}(e_i \otimes e_j + e_j \otimes e_i) \,\middle|\, 0 \le i < j \le d \right\} .$$

$$\dim \mathcal{H}_+ = \frac{d(d+1)}{2} = \binom{d+1}{2} \ ;$$

$$\text{basis of } \mathcal{H}_- : \quad \left\{ \frac{1}{\sqrt{2}}(e_i \otimes e_j - e_j \otimes e_i) \,\middle|\, 0 \le i < j \le d \right\} .$$

$$\dim \mathcal{H}_- = \frac{d(d-1)}{2} = \binom{d}{2} \ .$$

Werner states

The group $SU(d)$ acts irreducibly on $\mathcal{H}_+$ and on $\mathcal{H}_-$.
Hence the action of $SU(d)$ has commutant

$$\{ u \otimes u \,|\, u \in SU(d) \}' = \{ \lambda p_+ + \mu p_- \,|\, \lambda, \mu \in \mathbb{C} \} = \{F\}'' ,$$

and the minimal nonzero symmetric projections are

$$p_\pm := \frac{\mathbb{1} \pm F}{2} = \text{projection onto } \mathcal{H}_\pm \ .$$

The $SU(d)$-symmetric states (i.e. Werner states) are convex combinations of

$$\omega_\pm := \text{(anti-)symmetric state:} \quad x \mapsto \frac{\mathrm{tr} p_\pm x}{\mathrm{tr} p_\pm} = \frac{\mathrm{tr}\left(\frac{\mathbb{1} \pm F}{2} x\right)}{\dim \mathcal{H}_\pm} \ .$$

The Werner states are given by

$$\omega = \lambda \omega_+ + (1 - \lambda)\omega_-, \quad 0 \le \lambda \le 1 \ .$$

We note that a Werner state is fixed by specifying its value on the flip operator:

$$\omega(F) = \lambda - (1 - \lambda) = 2\lambda - 1 \ .$$

Entanglement of Werner states for $n = 2$

For $x \in M_d \otimes M_d$, let Tx denote its average over the group $SU(d)$:

$$Tx := \int_{SU(d)} (u \otimes u)^* x (u \otimes u)\, du\ ,$$

where du denotes the Haar measure on $SU(d)$.

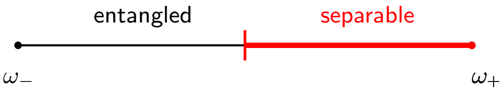
For states: $T^*\vartheta : x \mapsto \vartheta(Tx)$: projection of ϑ onto the Werner states, $T^*\vartheta$ coincides with ϑ on F.

Theorem

A Werner state ω on $M_d \otimes M_d$ is separable iff

$$\omega(F) \geq 0\ .$$

This the optimal Bell inequality for Werner states on two particles.



Proof.

For any pure product state $\psi \otimes \varphi \in \mathbb{C}^d \otimes \mathbb{C}^d$ the expectation of F is positive:

$$\langle \psi \otimes \varphi, F(\psi \otimes \varphi) \rangle = \langle \psi \otimes \varphi, \varphi \otimes \psi \rangle = \langle \psi, \varphi \rangle \langle \varphi, \psi \rangle = |\langle \psi, \varphi \rangle|^2 \geq 0 \; .$$

This inequality extends to all separable states by convexity.

Conversely, suppose $0 \leq \omega(F) \leq 1$ for some Werner state ω,
and choose unit vectors ψ, φ with

$$|\langle \psi, \varphi \rangle|^2 = \omega(F) \; .$$

Then the separable state

$$\sigma : x \mapsto \langle \psi \otimes \varphi, T(x) \psi \otimes \varphi \rangle = \int_{SU(d)} \langle (u \otimes u) \psi \otimes \varphi, x(u \otimes u) \psi \otimes \varphi \rangle \, du$$

is a Werner state, and coincides with ω on F.
Hence $\omega = \sigma$, which is separable. $\square$

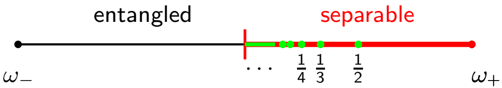
The chaotic state moves to the boundary as $d \to \infty$

Curious fact:

$$\lim_{d\to\infty} \tau_d \otimes \tau_d(F) = 0 \; .$$

Indeed,

$$\begin{aligned} \tau_d \otimes \tau_d(F) &= \frac{1}{d^2}\mathrm{tr}_d \otimes \mathrm{tr}_d(F) = \frac{1}{d^2}\sum_{i,j=1}^{d} \langle e_i \otimes e_j, F(e_i \otimes e_j)\rangle \\ &= \frac{1}{d^2}\sum_{i,j=1}^{d} \langle e_i \otimes e_j, e_j \otimes e_i\rangle = \frac{1}{d^2}\sum_{i,j=1}^{d} \delta_{ij} = \frac{1}{d} \; . \end{aligned}$$



General $n \in \mathbb{N}$: Schur-Weyl duality

On the Hilbert space

$$\mathcal{H} := \mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d \qquad (n \text{ times})$$

there are representations of two groups: S_n and $SU(d)$:

$$\begin{aligned} S_n \ni \sigma \quad &: \quad \pi(\sigma)\psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_n := \psi_{\sigma^{-1}(1)} \otimes \psi_{\sigma^{-1}(2)} \otimes \cdots \otimes \psi_{\sigma^{-1}(n)} \\ SU(d) \ni u \quad &: \quad \pi'(u)\psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_n := u\psi_1 \otimes u\psi_2 \otimes \cdots \otimes u\psi_n \end{aligned}$$

The classical Schur-Weyl duality theorem states that these two group actions do not only commute, but the algebras they generate are actually each other's commutant.

In particular they have the same center:

$$\mathcal{Z} := \mathcal{Z}(n,d) := \pi(S_n)' \cap \pi'(SU(d))' \; .$$

The minimal projections in this center cut both group representations into their irreducible components, and they are labeled by Young diagrams.

Indeed we have

$$\pi(\sigma)\pi'(u) \cong \bigoplus_{\lambda \vdash n} \pi_\lambda(\sigma) \otimes \pi'_\lambda(u) \; .$$

In particular

$$d^n = \sum_{\lambda \vdash n} d(\lambda) d'(\lambda) \; .$$

The group algebra of S_n

Let $\mathcal{A}_n$ denote the group algebra of S_n:

$$f : S_n \to \mathbb{C} \qquad \text{to be viewed as} \qquad \sum_{\sigma \in S_n} f(\sigma)\sigma \; .$$

Multiplication in $\mathcal{A}_n$ is convolution:

$$(f * g)(\sigma) = \sum_{\tau \in S_n} f(\tau) g(\tau^{-1}\sigma) \; .$$

The unit is δ_e, where e is the identity element of S_n.
Adjoint operation:

$$f^*(\sigma) = \overline{f(\sigma^{-1})} \; .$$

Every unitary representation of S_n automatically extends to a representation of $\mathcal{A}_n$.
In our case

$$\pi(f) : \quad \psi_1 \otimes \ldots \otimes \psi_n \mapsto \sum_{\sigma \in S_n} f(\sigma) \psi_{\sigma^{-1}(1)} \otimes \ldots \otimes \psi_{\sigma^{-1}n} \; .$$

The (left) regular representation of S_n

We let $f \in \mathcal{A}_n$ act on the Hilbert space $l^2(S_n)$ by convolution on the left:

$$h \mapsto f * h \ .$$

The trace is in this representation of a particularly simple form:

$$\mathrm{trreg}(f) := \sum_{\sigma \in S_n} \langle \delta_\sigma, f * \delta_\sigma \rangle = \sum_{\sigma \in S_n} (f * \delta_\sigma)(\sigma) = n! \cdot f(e) \ ,$$

and will be called the regular trace.

The normalized version $\tau\mathrm{reg} := \frac{1}{n!}\mathrm{trreg}$ is the regular trace state:

$$\tau\mathrm{reg} : \quad f \mapsto f(e) \ .$$

The center of the group algebra of S_n

$$\mathcal{Z}_n := \mathcal{A}_n \cap \mathcal{A}'_n \ .$$

We have $f \in \mathcal{Z}_n$ if and only if for all $\sigma, \tau \in S_n$: $f(\sigma\tau) = f(\tau\sigma)$: the center consists of the class functions. Hence

$$\begin{aligned} \dim \mathcal{Z}_n &= \#(\text{conjugacy classes of } S_n) \\ &= \#(\text{partitions of } n) =: \mathcal{P}(n) \ . \end{aligned}$$

On the other hand, since $\mathcal{Z}_n$ is an abelian matrix algebra, it must be of the form

$$\mathcal{Z}_n = \bigoplus_{\lambda \vdash n} \mathbb{C} p_\lambda$$

for some orthogonal set of minimal projections p_λ in the center. These can be labeled by Young diagrams.
The states on the center form a simplex with extreme points ω_λ given by

$$\omega_\lambda(p_\mu) = \delta_{\lambda\mu} \ .$$

minimal projections and irreducible representations

The center of the algebra $\mathcal{A}_n$ is spanned by the minimal projections:

$$p_\lambda(\sigma^{-1}) = \overline{p_\lambda(\sigma)}, \quad p_\lambda * p_\mu = \delta_{\lambda\mu} p_\lambda \quad \text{and} \quad \sum_{\lambda \vdash n} p_\lambda = \delta_e \ .$$

They cut the algebra $\mathcal{A} = \mathcal{A}_n$ into factors $p_\lambda \mathcal{A}$:

$$\mathcal{A} = \bigoplus_{\lambda \vdash n} p_\lambda \mathcal{A} \quad \simeq \quad \bigoplus_{\lambda \vdash n} M_{d(\lambda)} \otimes \mathbb{1}_{d(\lambda)} \ .$$

Hence

$$d(\lambda)^2 = \quad \mathrm{tr}(p_\lambda) \quad = \quad n! \cdot p_\lambda(e) \ .$$

The characters χ_λ and χ'_λ

The character $\chi_\lambda(\sigma)$ is the trace of σ in its irreducible representation π_λ.

$$\chi_\lambda(\sigma) := \mathrm{tr}\big(\pi_\lambda(\sigma)\big)$$
$$\chi'_\lambda(u) := \mathrm{tr}\big(\pi'_\lambda(u)\big) \ .$$

$\chi'(\lambda)$ is given by the Schur polynomials

$$\chi'_\lambda(\mathrm{diag}(x_1, \ldots, x_d)) = s_\lambda(x_1, \ldots, x_d) \ .$$

$\chi_\lambda(\sigma)$ is directly related to the projection operator p_λ:

$$\chi_\lambda(\sigma) = \frac{n!}{d(\lambda)} \cdot p_\lambda(\sigma) \ .$$

Young frames

The irreducible representations of S_n (and hence also the minimal central projections and the characters) are labelled by Young frames with n boxes:

$$\lambda = \ydiagram{4,3,3,1} \; .$$

(Hook length rule)

$$d(\lambda) = \frac{n!}{\prod \text{hook lengths}} \; .$$

For example:

$$d\left(\ydiagram{3,2}\right) = \frac{5!}{4 \times 3 \times 2} = 5 \quad \text{hook lengths:} \quad \begin{ytableau} 4 & 3 & 1 \\ 2 & 1 \end{ytableau} \; .$$

Young vectors

The irreducible representations of S_n (and hence also the minimal central projections and the characters) can be constructed from Young frames with n boxes, for example:

$$\lambda = \ydiagram{4,3,1} \; .$$

To λ we associate a unit vector $\psi_\lambda \in (\mathbb{C}^d)^{\otimes n}$:

$$\psi_\lambda := \psi_{\ydiagram{1,1,1}} \otimes \psi_{\ydiagram{1,1}} \otimes \psi_{\ydiagram{1,1}} \otimes \psi_{\ydiagram{1}} \; .$$

Here, $\psi_{\ydiagram{1,1,1,1}}$ (with height k) is the antisymmetric product

$$\varepsilon_1 \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_k \; .$$

in terms of the canonical basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d$ of $\mathbb{C}^d$.

Construction of the minimal projections

Theorem

The projection onto the linear span of the vectors

$$\pi(\sigma)\psi_\lambda \; , \qquad (\sigma \in \mathcal{S}_n)$$

is a minimal projection in the commutant

$$\pi(\mathcal{A}_n)' \; .$$

On this space π acts as an irreducible representation of S_n.
The projection onto the linear span of the vectors

$$\pi(\sigma)\pi'(u)\psi_\lambda \; , \qquad (\sigma \in \mathcal{S}_n, u \in SU(d))$$

is the image $\pi(p_\lambda)$ of the minimal projection $p_\lambda \in \mathcal{Z}$.
We have for $\lambda \neq \lambda'$,

$$\pi(p_\lambda)\pi(p_{\lambda'}) = 0 \; .$$

Moreover,

$$\sum_\lambda \pi(p_\lambda) = \mathbb{1}_{\mathcal{H}_{n,d}} \; .$$

Generalized Pauli exclusion principle

Theorem

Let $n, d \in \mathbb{N}$. Let λ denote a Young frame with n boxes. Then

$$\pi_{n,d}(p_\lambda) = 0 \quad \textit{iff} \quad \textit{height}(\lambda) > d \ .$$

Proof.

The Young vector ψ_λ fits into a d-eimensional space iff

$$\text{height}(\lambda) \leq d \ .$$

$\square$

For example, the symmetric subspace, having Young frame $\square\square\square\square$, is nonzero in $(\mathbb{C}^d)^{\otimes 4}$ for every one-particle dimension d, but, according to Pauli's exclusion principle, the antisymmetric subspace, with Young frame $\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}$, needs $d \geq 4$. Hence the above theorem generalizes this exclusion principle.

Symmetric Werner states on $\mathcal{B}\big((\mathbb{C}^d)^{\otimes n}\big)$

Observables (operators) on $\mathcal{H} := \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d$ can be 'twirled' and averaged:

$$\begin{aligned} Ta &:= \int_{SU(d)} (u \otimes \ldots \otimes u)^* \, a \, (u \otimes \ldots \otimes u) \, du \; ; \\ Ma &:= \frac{1}{n!} \sum_{\sigma \in S_n} \pi(\sigma)^* \, a \, \pi(\sigma) \; . \end{aligned}$$

Clearly, $Ta \in \pi'(SU(d))'$, and in the same way $Ma \in \pi(S_n)'$.
Hence $P := TM = MT$ projects onto the center $\pi(\mathcal{Z}_n)$,
Dually P^* takes a state ϑ, restricts it to the center, and then extends it to a symmetric Werner state on $\mathcal{B}(\mathcal{H})$:

$$(P^* \vartheta)(a) := \vartheta(Pa) \; .$$

Theorem (**Separability of symmetric Werner states)**

Let ϑ be a symmetric Werner state on $\mathcal{B}\big((\mathbb{C}^d)^{\otimes n}\big)$. Then ϑ is separable iff its restriction to $\mathcal{Z}$ lies in the convex hull of the restricted product states.

Conclusion: We must calculate the shadow of the product states!

The trace state

Theorem

The trace state on $M_d^{\otimes n}$ moves towards the regular trace on $\mathcal{Z}_n$ as $d \to \infty$.

Proof.

First we calculate:

$$\begin{aligned}
\mathrm{tr}_d^{\otimes n}(\pi(\sigma)) &= \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} \langle e_{i_1} \otimes \cdots \otimes e_{i_n}, \pi(\sigma)\, e_{i_1} \otimes \cdots \otimes e_{i_n} \rangle \\
&= \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} \delta_{i_1 i_{\sigma^{-1}(1)}} \cdots \delta_{i_n i_{\sigma^{-1}(n)}} \, . \\
&= d^{\#(\text{cycles of } \sigma)} \, .
\end{aligned}$$

since for every cycle one summation variable remains. Hence:

$$\tau_d^{\otimes n}(p_\lambda) = \frac{d(\lambda)}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \frac{1}{d^n} \mathrm{tr}_d^{\otimes n}(\pi(\sigma)) \xrightarrow{d \to \infty} \frac{d(\lambda)^2}{n!} = \tau_{\mathrm{reg}}(p_\lambda) \, .$$

$\square$

The shadow of the product states

The state space $\mathcal{S}(\mathcal{Z})$ of the center $\mathcal{Z}$ is a simplex whose corners are the states

$$\omega_\lambda : \quad p_{\lambda'} \mapsto \delta_{\lambda\lambda'} \ .$$

The product states throw their shadow on this simplex: the barycentric coordinates of the product state $\psi = \psi_1 \otimes \ldots \otimes \psi_n$ w.r.t. these ω_λ are given by

$$w_\psi(\lambda) := \langle \psi_1 \otimes \ldots \otimes \psi_n, \pi(p_\lambda)\, \psi_1 \otimes \ldots \otimes \psi_n \rangle \ .$$

We note that the the regular trace has coordinates:

$$w_{\mathrm{reg}}(\lambda) := \tau_{\mathrm{reg}}(p_\lambda) = \frac{d(\lambda)^2}{n!} \ .$$

Now here's our basic connection between entanglement and classical mathematics:

Theorem (Barycentric coordinates of a product state)

The coordinates of a product state $\psi_1 \otimes \ldots \otimes \psi_n$ are obtained by multiplying those of the regular trace with the normalized immanant of the Gram matrix of $\psi_1, \psi_2, \ldots, \psi_n$:

$$w_\psi(\lambda) = w_{\mathrm{reg}}(\lambda) \cdot \widetilde{\mathrm{Imm}}_\lambda \left(\left(\langle \psi_i, \psi_j \rangle \right)_{i,j=1}^n \right) \ .$$

Immanants of a matrix

Let A be an $n \times n$ matrix, and let λ be a Young frame with n boxes.
Then the immanant $\mathrm{Imm}_\lambda(A)$ of this matrix associated to λ is defined as

$$\mathrm{Imm}_\lambda(A) := \sum_{\sigma \in S_n} \chi_\lambda(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \ .$$

The normalized immanant $\tilde{\mathrm{Imm}}_\lambda(A)$ is defined so as to have $\tilde{\mathrm{Imm}}(\mathbb{1}) = 1$:

$$\tilde{\mathrm{Imm}}_\lambda(A) := \frac{\mathrm{Imm}_\lambda(A)}{d(\lambda)} \ .$$

Note the following well-known special cases:

$$\mathrm{Imm}_{\ydiagram{1,1,1,1,1,1,1}}(A) = \det(A) \quad \text{and} \quad \mathrm{Imm}_{\ydiagram{7}}(A) = \mathrm{per}(A) \ .$$

We mention the following inequalities: for all positive definite matrices A and all Young frames λ:

$$\det(A) \leq \tilde{\mathrm{Imm}}_\lambda(A) \leq \mathrm{per}(A) \ .$$

The first inequality was proved by Schur in 1918, the second was conjectured by Elliott Lieb in 1967, and is still open!

Calculation of coordinates

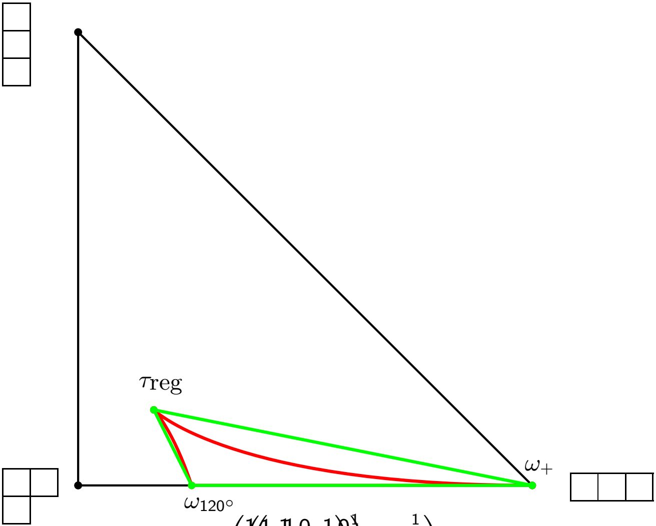
Proof.

This is not more than a concatenation of definitions connecting quantum information (entanglement) to algebra (immanants).
The weight of the extreme point ω_λ in the expansion of the pure product state $\psi = \psi_1 \otimes \ldots \otimes \psi_n$ is equal to

$$\begin{aligned}
w_\psi(\lambda) &= \langle \psi_1 \otimes \ldots \otimes \psi_n, \pi(p_\lambda)\, \psi_1 \otimes \ldots \otimes \psi_n \rangle \\
&= \frac{d(\lambda)}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \langle \psi_1 \otimes \ldots \otimes \psi_n, \pi(\sigma) \psi_1 \otimes \ldots \otimes \psi_n \rangle \\
&= \frac{d(\lambda)}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \langle \psi_1 \otimes \ldots \otimes \psi_n, \psi_{\sigma^{-1}(1)} \otimes \cdots \otimes \psi_{\sigma^{-1}(n)} \rangle \\
&= \frac{d(\lambda)}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{j=1}^{n} \langle \psi_j, \psi_{\sigma^{-1}(j)} \rangle \\
&= \frac{d(\lambda)}{n!} \mathrm{Imm}_\lambda(G(\psi)) = \frac{d(\lambda)^2}{n!} \tilde{\mathrm{I}}\mathrm{mm}_\lambda(G(\psi)) \, .
\end{aligned}$$

$\square$

The simplex $\mathcal{S}(\mathcal{Z}_n)$ for $n = 3$



The separable states

The set of product states!

$\omega_{120^\circ} = P(|\psi\rangle\langle\psi|)$ with $G(\psi) = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$;

Optimal Bell inequalities for $n = 3$

For $n = 3$ the separable region is a polytope, having a finite number (3) of extreme points.
We need only two linear ('Bell') inequalities in order to distinguish the separable from the entangled symmetric Werner states.

$$\omega(p_+ + 5p_-) \geq 1 \; ;$$
$$\omega(4p_+ + p_-) \geq 1 \; .$$

They correspond to the green lines in the figure.

Questions:

- Is the convex hull of the separable symmetric Werner states (the 'shadow') always a polytope?
- What is the general shape of this region?
- Does is grow or shrink with increasing n?

Shadow moves away from the antisymmetric state

Theorem

For all separable symmetric Werner states ω we have

$$\omega(p_-) \leq \frac{1}{n!}$$

with equality only for the regular trace state.

Proof.

The determinant of the Gram matrix of an n-tuple of unit vectors is equal to

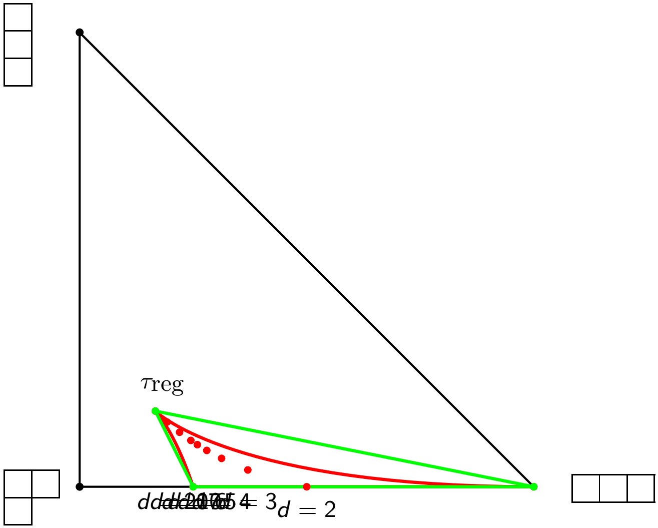
$$\begin{aligned} \det\big(\langle\psi_i,\psi_j\rangle\big) &= \det\Big(\sum_{k=1}^{n}\langle\psi_i,e_k\rangle\langle e_k,\psi_j\rangle\Big) \\ &= \Big|\det\big(\langle\psi_i,e_k\rangle\big)\Big|^2 = \mathrm{vol}(\psi_1,\psi_2,\ldots,\psi_n)^2 \leq 1 \ ; \end{aligned}$$

Hence

$$\langle\psi_1\otimes\ldots\otimes\psi_n, p_-\,\psi_1\otimes\ldots\otimes\psi_n\rangle \leq \tau_{\mathrm{reg}}(p_-) = \frac{1}{n!}\ .$$

$\square$

The trace state kreeps onto the peninsula as d increases



The Schur and Lieb inequalities

We have $2\mathcal{P}(n) - 3$ inequalities, which divide the state space $\mathcal{S}(\mathcal{Z}_n)$ into compartments, and claim the the shadow of the product states falls into one of them.

Schur's 1918 inequality implies that for all separables symmetric Werner states ω and all Young frames λ:

$$\omega(p_\lambda) \geq d(\lambda)^2 \, \omega(p_-) \ .$$

in particular we have the rather trivial inequality ω:

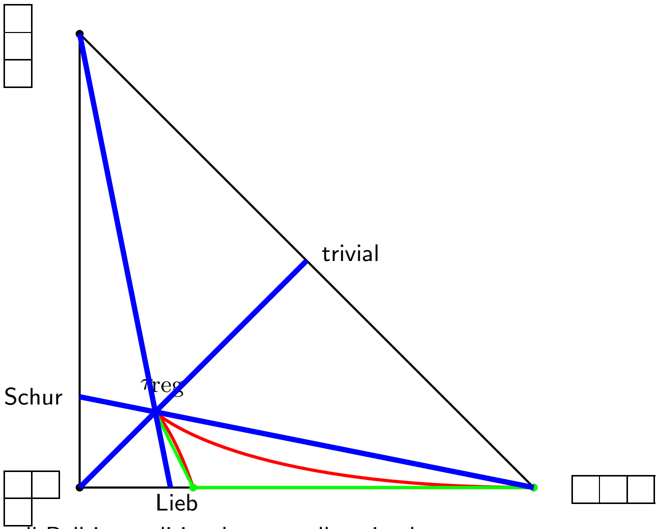
$$\omega(p_-) \leq \omega(p_+) \ .$$

Lieb's 1967 conjecture hopes that for all separable ω and all Young frames λ:

$$\omega(p_\lambda) \leq d(\lambda)^2 \, \omega(p_+) \ .$$

These are all Bell inequalities.

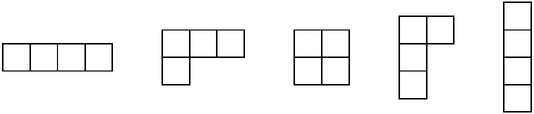
The immanant inequalities for $n = 3$ in a picture



These are all Bell inequalities, but not all optimal.

The separable region for $n = 4$

In the case $n = 4$ there are five Young diagrams:



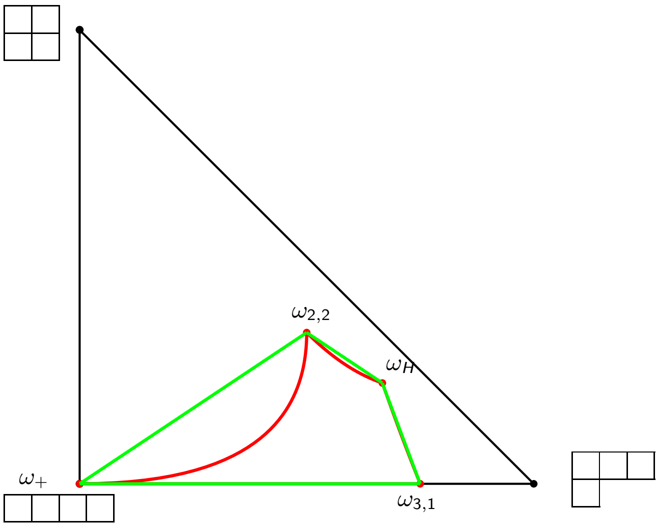
Theorem (Barrett, Hall, Loewy (1998) translated to quantum states)

The set of separable symmetric Werner states on $\mathcal{B}(\mathbb{C}^d)^{\otimes 4}$ is the convex hull of 7 extreme points. These extremal states are obtained by twirling and averaging 7 configurations of unit vectors in $\mathbb{C}^d$ (with $d \geq 4$ to fit all of them). These configurations are given by the Gram matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & i & i & -i \\ -i & \sqrt{3} & i & i \\ -i & -i & \sqrt{3} & -i \\ i & -i & i & \sqrt{3} \end{pmatrix}.$$

Four qubits



Hope crashed at $n = 5$

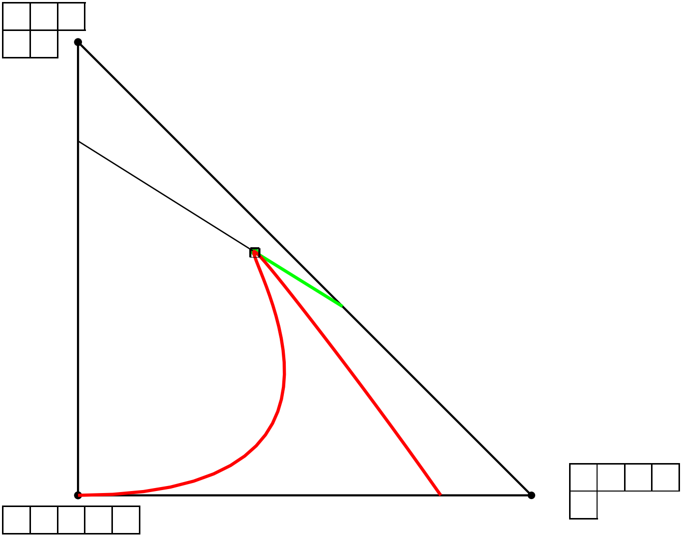
Our hope was, to prove that for all $n \in \mathbb{N}$ the separable symmetric Werner states would form a polytope.
However, this hope breaks down at $n = 5$:

Theorem (Barrett, Hall, Loewy (1999) translated)

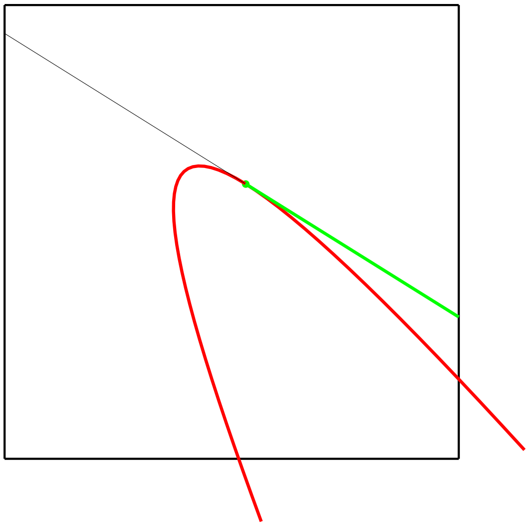
The set of all symmetric Werner separable states on $\mathcal{B}\big((\mathbb{C}^d)^{\otimes 5}\big)$ has an infinite number of extremal points.

In 1999 they showed that, already in the five qubit situation, the set of separable states on the center possesses a part that is bulging outward.

Proof that for $n = 5$ separables states do not form a polytope



Magnification



Extendability of symmetric Werner states

We say that $\mu \subset \lambda$ if for all $j \geq 1$ we have $\mu_j \leq \lambda_j$.
A Young frame μ and a frame λ are said to be adjacent if λ is obtained from μ by adding a single block.
By connecting such pairs a directed graph is obtained: the Young graph.
By the theory of representations of the permutation groups S_n with $n \in \mathbb{N}$, this is also the graph of restriction and induction of representations.
It follows that, for $\mu \subset \lambda$ with $\#\mu = k, \#\lambda = n$:

$$\omega_\lambda(p_\mu \otimes \mathbb{1}_{n-k}) = \frac{\#\{\text{paths } \phi \to \lambda \text{ via } \mu\}}{\#\{\text{paths } \phi \to \lambda\}} \ .$$

These are the barycentric coordinates of the restriction of ω_λ to $\mathcal{Z}_k \otimes \mathbb{1}_{n-k}$, i.e. the symmetric Werner state on k particles obtained by restriction of ω_λ.
Conversely, a state on $\mathcal{Z}_k$ is extendable to a symmetric Werner state on $\mathcal{H}^{\otimes n}$ iff it lies in the convex hull of such restrictions, where λ runs through the partitions of n.

Example: $k = 3$ and $n = 9$

Let λ and μ be the Young frames

$$\lambda = \ydiagram{3,3,3}\,, \quad \mu = \ydiagram{3}\,.$$

Then

$$X := \omega_\lambda(p_{\ydiagram{3}} \otimes \mathbb{1}_6) = \frac{\#\left\{\text{paths } \phi \to \ydiagram{3,3}\right\}}{\#\left\{\text{paths } \phi \to \lambda\right\}} = \frac{5}{42}\,,$$

and, by symmetry also

$$Y := \omega_\lambda(p_{\ydiagram{1,1,1}} \otimes \mathbb{1}_6) = \frac{5}{42}\,.$$

It follows that

$$Z := \omega_\lambda(p_{\ydiagram{2,1}} \otimes \mathbb{1}_6) = \frac{32}{42}\,.$$

Hence the restriction of ω_λ to 3 particles has coordinates $\frac{1}{42}(5, 5, 32)$.

Tensor power states on $\mathcal{Z}_k$

A particular kind of separable symmetric Werner states are the twirled tensor power states, described by their restriction to $\mathcal{Z}_k$ by

$$z \mapsto \rho^{\otimes k}(z) \ ,$$

where $z \in \mathcal{Z}_k$ and ρ is a state on M_d.
Clearly, these states are separable Werner states.
But not all separables Werner states are of this form:

Lemma

Tensor power states take positive values on the permutations $\pi(\sigma)$ with $\sigma \in S_n$, but there exist separable Werner states taking negative values on these.

Proof.

Let ρ have diagonal density matrix with entries $(x_1, \ldots, x_d)$.
Let σ be a permutation with cycle lengths $s_1, \ldots, s_l$. Then

$$\rho^{\otimes k}(\pi(\sigma)) = (x_1^{s_1} + \ldots + x_d^{s_1}) \cdot (x_1^{s_2} + \ldots + x_d^{s_2}) \cdots (x_1^{s_l} + \ldots + x_d^{s_l}) \geq 0 \ .$$

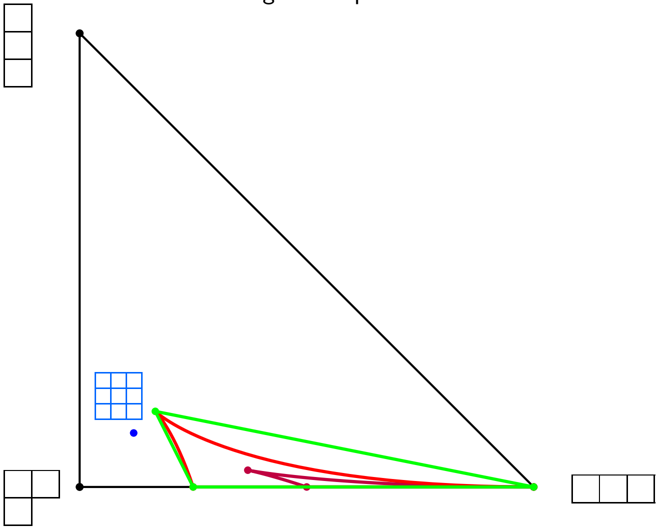
However, the pure product state $\psi_1 \otimes \psi_2 \otimes \psi_3$ on $\mathcal{Z}_3$ with $\langle \psi_i, \psi_j \rangle = -\frac{1}{2}$ for $i \neq j$ has

$$\langle \psi, \pi(123)\psi \rangle = -\frac{1}{8} \ .$$

□

Tensor power states on $\mathcal{Z}_3$ with $d = 3$

Three qutrits can be in the following tensor power states:

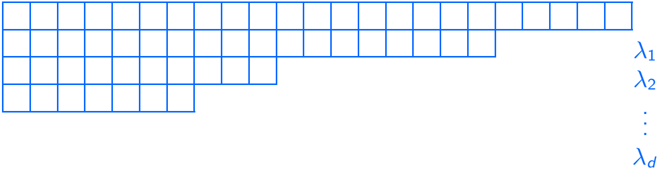


A de Finetti Theorem for Symmetric Werner States

(Christandl, König, Mitchison, Renner, 2008)

Theorem

Let λ be a Young frame of height d and a large number n of blocks:



Let ρ_λ denote the state on M_d with density matrix

$$\frac{1}{n}\begin{pmatrix} \lambda_1 & & & \emptyset \\ & \lambda_2 & & \\ & & \ddots & \\ \emptyset & & & \lambda_d \end{pmatrix}.$$

Then for all $z \in \mathcal{Z}_k \subset M_d^{\otimes k}$ with $||z|| \leq 1$:

$$|\omega_\lambda(z \otimes \mathbb{1}_{n-k}) - \rho_\lambda^{\otimes k}(z)| \leq \frac{3}{4}\frac{k(k-1)}{\lambda_d} + O\left(\frac{k^4}{\lambda_d^2}\right), \quad (n \to \infty).$$

Spectacular Result of Okounkov and Olshanski

Theorem

$$\frac{\#\{paths\ \mu \to \lambda\}}{\#\{paths\ \varphi \to \lambda\}} = \frac{s^*_\mu(\lambda_1 \ldots, \lambda_d)}{n(n-1)\cdots(n-k+1)} \ ,$$

where the shifted Schur functions s^*_μ *are give by*

$$s_\mu(x_1, \ldots, x_d) := \sum_{T \downarrow \mu} \prod_{(i,j) \in T} \left(x_{T(i,j)} - (i-j)\right) \ .$$

Sketch of the proof of Christandl, König, Mitchison, and Renner

Proof.

It then follows that

$$
\begin{aligned}
\omega_\lambda(p_\mu \otimes \mathbb{1}_{n-k}) &= \frac{\#\{\text{paths } \phi \to \lambda \text{ via } \mu\}}{\#\{\text{paths } \phi \to \lambda\}} \\
&= d(\mu) \cdot \frac{s^*_\mu(\lambda_1 \dots, \lambda_d)}{n(n-1)\cdots(n-k+1)} \\
&= d(\mu) \cdot \sum_{T \downarrow \mu} \frac{\prod_{(i,j)\in\mu}\big(x_{T(i,j)} - (i-j)\big)}{n(n-1)\cdots(n-k+1)} \\
&\overset{n\to\infty}{\longrightarrow} d(\mu) \cdot \sum_{T \downarrow \mu} \prod_{(i,j)\in\mu} \frac{\lambda_{T(i,j)}}{n} \\
&= s_\mu\big(\frac{\lambda_1}{n}, \dots, \frac{\lambda_d}{n}\big) = \mathrm{tr}\big(\pi(p_\mu)\pi'(\rho_\lambda)\big) = \rho_\lambda^{\otimes n}\big(\pi(p_\mu)\big)\ .
\end{aligned}
$$

□

Extendable states from $n = 3$ to $n = 12$