1 Cartan’s first fundamental theorem.

Second lecture on Singer and Sternberg’s 1965 paper [3], by Bas Janssens.

1.1 Introduction

Let \( M \) be a smooth connected manifold and let \( \text{Vec} \) be the sheaf of smooth vector fields on \( M \). Let \( \mathcal{L} \subseteq \text{Vec} \) be a sheaf of Lie algebras of vector fields, that is, a subsheaf of \( \text{Vec} \) that is closed under the Lie bracket. For \( x \in M \), let \( \mathcal{L}_x \) be the Lie algebra of germs of sections of \( \mathcal{L} \) around \( x \). It is filtered by the subalgebras \( \mathcal{L}_{x,k} \) of germs that vanish to order \( k \) (where we set \( \mathcal{L}_{x,k} = \mathcal{L}_x \) for \( k < 0 \)).

**Definition 1.1.** We define the formal Lie algebra of \( \mathcal{L} \) at \( x \) by

\[
\mathfrak{L}_x := \lim_{\leftarrow k} \mathcal{L}_{k|x},
\]

where \( \mathcal{L}_{k|x} := \mathcal{L}_x/\mathcal{L}_{x,k} \) is the space of \( k \)-jets of sections of \( \mathcal{L} \) at \( x \).

It is filtered by the subalgebras \( \mathfrak{L}_{x,k} := \lim_{\leftarrow n} \mathcal{L}_{x,k}/\mathcal{L}_{x,n} \) of formal jets that vanish to order \( k \). If we endow \( \mathfrak{L}_x \) with the inverse limit topology induced by the norm topology of the (finite dimensional) spaces \( \mathcal{L}_{k|x} \), then \( \mathfrak{L}_x \) becomes a Fréchet Lie algebra. For every open neighbourhood \( U \) of \( x \), the Lie algebra \( \mathcal{L}(U) \) is a (not necessarily closed) locally convex subalgebra of \( \text{Vec}(U) \), and the evaluation \( \text{ev}_x : \mathcal{L}(U) \to \mathfrak{L}_x \) is a morphism of locally convex Lie algebras.

**Definition 1.2.** We define \( \mathfrak{g}_x \) to be the associated graded Lie algebra of \( \mathfrak{L}_x \).

More explicitly, if we define \( \mathfrak{g}_x^k \) to be the kernel of the map \( \mathcal{L}_{k+1|x} \to \mathcal{L}_{k|x} \), that is, the space of \( k+1 \)-symbols of sections of \( \mathcal{L} \) at \( x \), then we have \( \mathfrak{g}_x^k = L_{x,k}/L_{x,k+1} \) (in particular, \( \mathfrak{g}_x^k = \{0\} \) for \( k < -1 \)), so

\[
\mathfrak{g}_x = \prod_{k=-1}^{\infty} \mathfrak{g}_x^k.
\]

With respect to the Lie algebra and topology inherited from \( \mathfrak{L}_x \), the associated graded Lie algebra \( \mathfrak{g}_x \) is a graded Fréchet Lie algebra.

An important example is obtained by taking \( \mathcal{L} = \text{Vec} \). Then \( \mathfrak{L}_x \) is \( J_x^\infty := \lim_{k \to \infty} J_x^k(TM) \), the inverse limit of the \( k \)-jets at \( x \) of sections of the tangent bundle. Note that \( J^\infty \to M \) is a locally trivial bundle of Fréchet Lie algebras.

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We denote the associated graded Lie algebra $\mathfrak{g}_x$ by $S_b_x$, the inverse limit over the spaces $S^k_b = \text{Ker}(J^{k+1}(TM) \to J^k(TM))$, equal to $\text{Vec}^x_k/\text{Vec}^x_{k+1}$, of symbols of order $k + 1$. Note that $S^k_b$ can be canonically identified with $S^{k+1}(T^*_x M) \otimes T_x M$, and that this identification is a bundle isomorphism. We thus have

$$S_b = \prod_{k=1}^{\infty} S^{k+1}(T^*_x M) \otimes TM$$

as a locally trivial bundle of graded Fréchet Lie algebras over $M$.

### 1.2 Some relevant algebra

In this section, we will see that the fact that $L_x$ is subalgebra of $S_b_x$ imposes severe restrictions on the algebraic structure of $L_x$.

The coadjoint representation of $S_b_x$ on its continuous dual

$$S_b^* = \bigoplus_{k=1}^{\infty} S^{k+1}(T^*_x M) \otimes T^*_x M$$

yields an action of the universal enveloping algebra $\mathcal{U}(S_b_x)$ on $S_b^*$. The inclusion of the abelian Lie algebra $T_x M = S_b^{-1}$ into $S_b_x$ yields an inclusion of $\mathcal{U}(S_b^{-1}) = S(T_x M)$ into $\mathcal{U}(S_b_x)$, hence an action of $S(T_x M)$ on $S_b^*$.

**Proposition 1.3.** The coadjoint action of $S(T_x M) \subseteq \mathcal{U}(S_b_x)$ on the continuous dual $S_b^* = S(T_x M) \otimes T^*_x M$ is induced by the symmetric tensor product $\vee: S(T_x M) \times S(T_x M) \to S(T_x M)$.

**Proof.** For $v \in T_x M$, the adjoint action $\text{ad}_v: S^k_b \to S^{k-1}_b$ is given on $\gamma \otimes w \in S^{k+1}(T^*_x M) \otimes T_x M$ by $\text{ad}_v(\gamma \otimes w) = (i_v \gamma) \otimes w$. The dual of the annihilation operator $i_v$ on $S(T^*_x M) := \bigotimes_{n=0}^{\infty} S^1(T^*_x M)$ is the creation operator $u \mapsto v \vee u$ on $S(T_x M)$. Since $\mathcal{U}(S_b^{-1})$ is generated by $T_x M$, the proposition follows.

**Corollary 1.4.** The annihilator $\text{ann}(\mathfrak{g}_x)$ of $\mathfrak{g}_x$ in $S_b^*$ is a $S(\mathfrak{g}_x^{-1})$-module.

**Proof.** For $v \in \mathfrak{g}_x^{-1} \subseteq T_x M$, we have $\text{ad}_v(\mathfrak{g}) \subseteq \mathfrak{g}$. The action of $\text{ad}_v^*$ on $S_b^*$ therefore preserves the subspace $\text{ann}(\mathfrak{g}_x)$. 

We call $\mathcal{L}$ transitive at $x \in M$ if $\mathfrak{g}_x^{-1} = T_x M$. Combining Corollary 1.4 with the Hilbert Basis Theorem and the Artin-Rees Lemma, we obtain the following theorem.

**Theorem 1.5.** If $\mathcal{L}$ is transitive at $x \in M$, then there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, the space $\mathfrak{g}_x^k \subseteq S^{k+1}(T_x^* M) \otimes T_x M$ is determined by $\mathfrak{g}_x^{k-1} \subseteq S^k(T_x^* M) \otimes T_x M$ by

$$\mathfrak{g}_x^k = (T_x^* M \otimes \mathfrak{g}_x^{k-1}) \cap (S^{k+1}(T_x^* M) \otimes T_x M) .$$

**Proof.** Since $\mathfrak{g}_x^{-1} = T_x M$, Corollary 1.4 implies that $\text{ann}(\mathfrak{g}_x)$ is a $S(T_x M)$-submodule of the finitely generated $S(T_x M)$-module $S_b^* = S(T_x M) \otimes T_x^* M$.

By the Hilbert Basis Theorem, $\text{ann}(\mathfrak{g}_x)$ is finitely generated itself, and the Artin-Rees Lemma then implies that the filtration of $\text{ann}(\mathfrak{g}_x)$ is essentially $m$-adic for
the maximal ideal \( m = \bigoplus_{i=1}^\infty S(T_x M) \) of \( S(T_x M) \). Since \( S(T_x M) \) and \( \mathfrak{g}_x \) are graded, there exists a \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \), we have

\[
\text{ann}(\mathfrak{g}^k) = T_x M \lor \text{ann}(\mathfrak{g}^{k-1}).
\] (1)

This implies that for sufficiently large values of \( k \), the space \( \mathfrak{g}_x^{k-1} \) determines \( \mathfrak{g}^k \). Dualising equation (1), we calculate

\[
\mathfrak{g}_x^k = \text{ann}(T_x M \otimes \text{ann}(\mathfrak{g}_x^{k-1})) \cap S^{k+1}(T_x^* M) \otimes T_x M
\]

\[
= T_x^* M \otimes \mathfrak{g}_x^{k-1} \cap S^{k+1}(T_x^* M) \otimes T_x M,
\]

where \( \text{ann}(T_x M \otimes \text{ann}(\mathfrak{g}_x^{k-1})) \) is the annihilator of \( T_x M \otimes \text{ann}(\mathfrak{g}_x^{k-1}) \) inside the tensor algebra \( T^{k+1}(T_x^* M) \otimes T_x M \).

In order to formulate this more succinctly, we introduce the notion of a prolongation.

**Definition 1.6.** The *prolongation* of a vector space \( U \subseteq \text{Hom}(V, W) \) is the subspace \( U^{(1)} \subseteq \text{Hom}(V, U) \) defined by

\[ U^{(1)} = \{ T \in \text{Hom}(V, U) ; T_u(v) = T_v(u) \lor u, v \in V \}. \]

If we identify \( \mathfrak{g}_x^{k-1} \subseteq S^k(T_x^* M) \otimes T_x M \) as a subspace of \( \text{Hom}(T_x M, S^{k-1}(T_x^* M) \otimes T_x M) \), then Theorem 1.5 can be reformulated as follows: if \( \mathcal{L} \) is transitive at \( x \in M \), then for sufficiently large values of \( k \), we have \( \mathfrak{g}_x^{k} = (\mathfrak{g}_x^{k-1})^{(1)} \). In this language, we obtain the following corollary from the proof of Theorem 1.5.

**Corollary 1.7.** If \( \mathcal{L} \) is transitive at \( x \in M \), then \( \mathfrak{g}_x^{k} \subseteq (\mathfrak{g}_x^{k-1})^{(1)} \) for all \( k \in \mathbb{N} \).

**Proof.** By Corollary 1.4, \( \text{ann}(\mathfrak{g}_x) \) is a \( S(T_x M) \)-module, implying \( \text{ann}(\mathfrak{g}_x^k) \supseteq T_x^* M \lor \mathfrak{g}_x^{k-1} \) for all \( k \in \mathbb{N} \). Repeating the dualisation at the end of the proof of Theorem 1.5, we obtain \( \mathfrak{g}_x^{k} \subseteq (T_x^* M \otimes \mathfrak{g}_x^{k-1}) \cap (S^{k+1}(T_x^* M) \otimes T_x M) \).

We can drop the requirement that \( \mathcal{L} \) be transitive at \( x \in M \) if we assume the following property:

**Definition 1.8.** The graded Lie algebra \( \mathfrak{g}_x \) is a *tower of tableaux starting at* \( k_0 \) if \( \mathfrak{g}_x^{r+1} \subseteq (\mathfrak{g}_x^r)^{(1)} \) for all \( r \geq k_0 \).

Note that every transitive \( \mathfrak{g}_x \) is a tower of tableaux starting at 0 by the previous corollary. If \( \mathcal{L} \) is defined by a regular PDE of order \( k_0 \), then it is a tower of tableaux starting at \( k_0 \).

**Corollary 1.9.** Let \( \mathcal{L} \) be such that \( \mathfrak{g}_x \) is a tower of tableaux starting at \( k_0 \). Then there exists a \( k \geq k_0 \) such that \( \mathfrak{g}_x^{r+1} = (\mathfrak{g}_x^r)^{(1)} \) for all \( r \geq k \).

**Proof.** The assumption that \( \mathfrak{g}_x^{r+1} \) be contained in \( (\mathfrak{g}_x^r)^{(1)} \) for all \( r \geq k_0 \) is equivalent to \( \bigoplus_{r=k_0}^\infty \text{ann}(\mathfrak{g}_x^r) \) being an \( S(T_x M) \)-submodule of \( S_{\mathcal{L}}^* \). The proof of Theorem 1.5, mutatis mutandis, then yields the required result.

**Definition 1.10.** We will say that \( \mathcal{L} \) is of order \( k \) if \( \mathfrak{g}_x^{r+1} = (\mathfrak{g}_x^r)^{(1)} \) for all \( r \geq k \).
1.3 Symmetries of a singular distribution

The $k^{\text{th}}$ order frame bundle $F^k(M)$ is the manifold of $k$-jets at zero of local diffeomorphisms $\psi$ from $U \subseteq \mathbb{R}^n$ to $\psi(U) \subseteq M$. Equipped with the projection $\pi: F^k(M) \to M$ defined by $\pi(j_0^k \psi) := \psi(0)$, it becomes a principal fibre bundle with structure group $\text{GL}^k(n)$, the group of $k$-jets of diffeomorphisms of $\mathbb{R}^n$ that fix $0$.

It carries an action by bundle automorphisms of the diffeomorphism group $\text{Diff}(M)$, defined by $\phi: j_0^k \psi \mapsto j_0^k(\phi \circ \psi)$. This yields a Lie algebra homomorphism $F_k: \text{Vec}(M) \to \text{Vec}(F^k(M))$. Any sheaf $\mathcal{L}$ of Lie algebras of vector fields on $M$ therefore gives rise to a sheaf $F_k \mathcal{L}$ of Lie algebras of vector fields on $F^k(M)$. This in turn gives rise to the (singular) distribution $\Delta^k \subseteq TF^k M$ of values of $F_k \mathcal{L}$.

Conversely, given a (singular) distribution $\Delta \subseteq TJ^k(M, M)$, we define the sheaf $\mathcal{L}_\Delta$ of infinitesimal symmetries of $\Delta$ by

$$\mathcal{L}_\Delta(U) := \{v \in \text{Vec}(U); \text{Im}(F_k(v)) \subseteq \Delta\}.$$  

Clearly, $\mathcal{L}$ is always a subsheaf of the sheaf of infinitesimal symmetries of the distribution $\Delta^k$ obtained from it, $\mathcal{L} \subseteq \mathcal{L}_\Delta^k$.

Let $J^k(M, M) \rightrightarrows M$ be the groupoid of $k$-jets of local diffeomorphisms of $M$. Since this is a transitive groupoid with a canonical left action on $F^k(M)$ (defined by $j^k \psi: j_0^k \psi \mapsto j_0^k(\phi \circ \psi)$), a choice $f_x$ of $k$-frame yields an identification of $F^k(M) \to M$ with the source fibre of $J^k(M, M) \rightrightarrows M$ over $x$, and of $J^k(M, M) \rightrightarrows M$ with the gauge groupoid $F^k(M) \times F^k(M)/\text{GL}^k(n) \rightrightarrows M$.

The action of $\text{Diff}(M)$ on $F^k(M)$ factors through the canonical splitting homomorphism

$$\Sigma: \text{Diff}(M) \to \text{Bis}(J^k(M, M)); \Sigma(\phi)_x := j^k_x \phi$$

into the group $\text{Bis}(J^k(M, M))$ of bisections. We identify $J^k(TM) \to M$ with the Lie algebroid of the groupoid $J^k(M, M)$ (that is, the pull back by the identity $e: M \to J^k(M, M)$ of the kernel $T^s J^k(M, M)$ of the differential $s_*: TJ^k(M, M) \to TM$ of the source map $s$). Then the Lie algebra homomorphism $\text{Vec}(M) \to \Gamma(J^k(TM))$ induced by the splitting homomorphism $\Sigma$ is precisely the map $v \mapsto j^k(v)$. This shows that the distribution $\Delta^k$ on $F^k(M)$ is the image of $L_x^k \subseteq J^k(TM)$ under the Lie algebroid action $\pi^* J^k_x \to TF^k(M)$. In particular, $F^k(v)_{f_x}$ is in $\Delta_{f_x}^k$ if and only if $j^k_x v$ is in $L_{f_x}^k$.

We will use this in following lemma, which says that under mild conditions on $\mathcal{L}$, the sheaves $\mathcal{L}$ and $\mathcal{L}_{\Delta^k}$ are in fact identical.

**Lemma 1.11.** Let $\mathcal{L}$ be a sheaf of Lie algebras of vector fields such that $\mathfrak{g}_x$ is a tower of tableaux starting at $k_0$ for all $x \in M$. Suppose that the order of $L_x$ is bounded by $k$ on $M$. Suppose also that $\mathfrak{g}_{\Delta^k}$ is a tower of tableaux starting at $k$. Then we have $L = L_{\Delta^k}$.

If, moreover, $\mathcal{L}$ is determined by $L$ in the sense that every $v \in \text{Vec}(U)$ with $j^\infty_x(v) \in L_x$ for all $x \in U$ belongs to $\mathcal{L}(U)$, then we have $\mathcal{L} = \mathcal{L}_{\Delta^k}$.

**Proof.** Since $\mathcal{L}$ is contained in $L_{\Delta^k}$, we have $L \subseteq L_{\Delta^k}$. Because $v$ is in $\mathcal{L}_{\Delta^k}(U)$ if and only if $F^k(v)_{f_x}$ is in $\Delta_{f_x}^k$ for all $f_x \in F^k(M)$, which in turn is the case if and only if $j^k_x(v) \in L_{f_x}^k$ for all $x \in U$, we have $L^k = L_{\Delta^k}^k$. In particular, $\mathfrak{g}^k = \mathfrak{g}_{\Delta^k}^k$. 

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for all \( r \leq k \). Now \( \mathfrak{g}^{k+r}_{\Delta^k} \subseteq (\mathfrak{g}^{k}_{\Delta^k})^{(r)} \) for all \( r > 0 \) by assumption. We thus find
\[
\mathfrak{g}^{k+r}_{\Delta^k} \subseteq (\mathfrak{g}^{k}_{\Delta^k})^{(r)} = (\mathfrak{g}^{k})^{(r)} = \mathfrak{g}^{k+r},
\]
the last equality following from the fact that \( \mathfrak{g} \) is of order at most \( k \). Since the opposite inclusion is clear, we have \( \mathfrak{g}^{k}_{\Delta^k} = \mathfrak{g}^k \) for all \( r \). From this and from \( L \subseteq L_{\Delta^k} \), it follows that \( L = L_{\Delta^k} \).

It remains to prove the last statement. We have already seen that \( \mathcal{L} \subseteq \mathcal{L}_{\Delta^k} \) and \( L = L_{\Delta^k} \). Now every \( v \in \mathcal{L}_{\Delta^k}(U) \) satisfies \( j^\infty_x(v) \in (L_{\Delta^k})_x = L_x \) for all \( x \in M \), hence lies in \( \mathcal{L}(U) \). Thus \( \mathcal{L}_{\Delta^k} \subseteq \mathcal{L} \) and the sheaves are equal. \( \square \)

The problem of realising \( \mathcal{L} \) as the sheaf of infinitesimal symmetries of a finite dimensional geometric object is thus essentially equivalent to the problem of integrating the singular foliation \( \Delta^k \).

By the Stefan-Sussman theorem \([4, \text{Corollary 1}]\), the singular distribution \( \Delta^k \) is integrable (through every point in \( J^k(M,M) \) passes an integral manifold of \( \Delta^k \)) if and only if the pushforward \( \exp(tX)_*: L^k_g \to L^k_y \) of the local flow \( \exp(tX) \) along \( X \in \mathcal{L}(U) \) is an isomorphism for all \( x \in \text{Dom}(\exp(tX)) \) and \( y = \exp(tX)(x) \).

The assumption that \( \mathfrak{g}_{\Delta^k} \) be a tower of tableaux is in this context a plausible assumption; if \( \Delta^k \) is sufficiently regular, it follows from the fact that \( \mathcal{L}_{\Delta^k} \) is defined by a PDE of order \( k \).

**Example 1.12.** Let \( \mathcal{L} \) be the sheaf of vector fields on \( \mathbb{R}^n \) that vanish at 0. Then one readily checks that \( \mathcal{L} \) satisfies the conditions of Lemma 1.11 for \( k = 0 \). The distribution \( \Delta^0 \) on \( F^0(M) = M \) is given by \( T_x \mathbb{R}^n \) for \( x \neq 0 \) and by \( \{0\} \) for \( x = 0 \). It is clearly integrable and yields the singular foliation of \( \mathbb{R}^n \) into 0 and \( \mathbb{R}^n - 0 \). Lemma 1.11 thus constructs a geometric object (the division of \( \mathbb{R}^n \) into \( \{0\} \) and \( \mathbb{R}^n - \{0\} \)) from the sheaf \( \mathcal{L} \), and yields the somewhat tautological statement that every vector field with \( j^0 v|_0 = 0 \) belongs to \( \mathcal{L} \).

**Example 1.13.** Let \( \mathcal{L} \) be the sheaf of vector fields on \( \mathbb{R}^n \) that vanish on the \( x^1 \)-axis \( \ell \). Then \( \mathfrak{g}_x = Sb_x \) for \( x \notin \ell \) and \( \mathfrak{g}_x = \sum_{i=2}^n x_i \cdot Sb_x \) for \( x \in \ell \). Hence \( (\mathfrak{g}^k)^{(i)} \) is strictly smaller than \( \mathfrak{g}^{k+1} \) for all \( k \), and Lemma 1.11 does not apply. Nonetheless, the distribution \( \Delta^0 \) is well defined and integrable, and the foliation of \( \mathbb{R}^n \) into \( \mathbb{R}^n - \ell \) and the points \( \{p\} \in \ell \) determines the sheaf.

### 1.4 Lie Algebra Sheaves (LAS) of order \( k \)

We streamline the process of applying Lemma 1.11 by imposing conditions on \( \mathcal{L} \) that insure regularity of \( \Delta^k \). We call a sheaf of Lie algebras regular of order \( k \) if \( L^k \to M \) is a smooth vector bundle.

**Definition 1.14.** We call a sheaf \( \mathcal{L} \) of Lie algebras of vector fields a **Lie Algebra Sheaf** (LAS) of order \( k \) if it is regular of order \( k \), if \( \mathfrak{g}_x \) is a tower of tableaux starting at \( k_0 \leq k \) and the order of \( L_x \) is bounded by \( k \) on \( M \), and if \( \mathcal{L} \) is determined by \( L \) in the sense that \( j^\infty_x v \in L_x \) for all \( x \in U \) implies \( v \in \mathcal{L}(U) \).

**Remark 1.15.** Every sheaf of Lie algebras of vector fields defined by a regular, linear PDE of order \( k_0 \) is a LAS of order \( k \) for some \( k \geq k_0 \).

**Remark 1.16.** Part of the definition of a LAS in the sense of Singer-Sternberg [3, Def. 1.8] is regularity of order 0. Our notion of a LAS of order \( k \) is therefore less restrictive.
The following theorem, which is essentially Cartan’s First Fundamental Theorem, is a reformulation of Lemma 1.11 for LAS of order \( k \).

**Theorem 1.17** (Cartan I for LAS of order \( k \)). For every LAS of order \( k \), there exists a Lie groupoid \( \mathcal{G}_L^k \rtimes M \) with a locally free action

\[
a: \mathcal{G}_L^k \times \pi F^k(M) \to F^k(M)
\]

such that \( \mathcal{L} \) is the sheaf of symmetries of this action, in the sense that \( \mathcal{L}(U) \) is the Lie algebra of all \( v \in \text{Vec}(M) \) such that the vector field \( F^k(v) \) on \( F^k(M) \) is parallel to the \( \mathcal{G}_L^k \)-orbits.

In other words: every LAS of order \( k \) is the sheaf of infinitesimal symmetries of a locally free groupoid action on the \( k^{th} \) order frame bundle.

**Proof.** The fact that \( L^k \to M \) is a smooth vector bundle implies that \( \Delta^k \) is a regular foliation. In particular, \( \mathcal{L}_{\Delta^k} \) is defined by a PDE of order \( k \), so that \( g_{\Delta^k} \) is a tower of tableaux starting at \( k \). It follows from Lemma 1.11 that \( \mathcal{L} = \mathcal{L}_{\Delta^k} \).

Since \( L^k \) is smooth and \( \mathcal{L} \) is closed under the Lie bracket, \( L^k \to M \) is a Lie subalgebroid of the integrable algebroid \( J^k(TM) \). By Prop. 3.4 and 3.5 in [1], \( L^k \) then integrates to a Lie groupoid \( \mathcal{G}_L^k \rtimes M \) with an immersive morphism \( \iota: \mathcal{G}_L^k \to J^k(M,M) \) of Lie groupoids. (In general, this immersion will be neither injective nor closed [2].) The free action of \( J^k(M,M) \) on \( F^k(M) \) then yields a locally free action of \( \mathcal{G}_L^k \) on \( F^k(M) \), and \( v \) is in \( \mathcal{L}_{\Delta^k}(U) \) if and only if \( F^k(v) \) is tangent to the orbits.

The following is a simple example of a LAS of order 1 which is not a LAS in the sense of Singer-Sternberg. This shows that Theorem 1.17 applies to a trictly wider class of sheaves than the ones in [3].

**Example 1.18.** Let \( \mathcal{L} \) be the sheaf of Lie algebras of vector fields on \( M = \mathbb{R}^2 \) defined by letting \( \mathcal{L}(U) := \mathbb{R} \cdot v|_U \) for \( v := x_1 \partial_{x_2} - x_2 \partial_{x_1} \). If we identify \( J^k_x \) with \( \prod_{k=1}^{\infty} \mathfrak{S}^{k-1}((\mathbb{R}^2)^*) \otimes \mathbb{R}^2 \), we obtain

\[
L((u_1, u_2)) = \mathbb{R} \cdot ((u_1 \partial_{x_2} - u_2 \partial_{x_1}) \oplus (dx_1 \otimes \partial_{x_2} - dx_2 \otimes \partial_{x_1})).
\]

Note that \( L^1 \to M \) is a smooth bundle, as are all the \( L^k \) with \( k \geq 1 \) and the bundle of Fréchet spaces \( L \to M \). The sheaf \( \mathcal{L} \) is a LAS of order 1 because, moreover, \( g_{\Delta}^k = 0 \) for \( k \geq 1 \).

Note, however, that none of the maps \( L^0 \to M, \ g^{-1} \to M \) and \( g^0 \to M \) have constant rank, so \( \mathcal{L} \) is not regular of order 0. The groupoid \( \mathcal{G}_L^1 \) integrating the Lie algebroid \( L^1 \to M \) is easily seen to be the action groupoid \( \mathbb{R}^2 \times \mathbb{S}^1 \) with the obvious action on the frame bundle \( F^1(\mathbb{R}^2) \).

This is the general situation for sheaves of Lie algebras that come from group actions with finite order fixed points.

**Corollary 1.19** (Symmetries of a group action). Let \( G \) be a connected Lie group, \( G \curvearrowright M \) a Lie group action, and \( \xi: \mathfrak{g} \to \text{Vec}(M) \) the associated Lie algebra morphism. Suppose that for every nonzero \( X \in \mathfrak{g} \), all fixed points of \( \xi_X \) are of order \( \leq k_0 \). Then \( \mathcal{L}(U) := \{ \xi_X|_U : X \in \mathfrak{g} \} \) is a LAS of order \( k_0 + 1 \). The corresponding groupoid \( G^k_L \) is the action groupoid \( G \times M \rtimes M \) with the obvious action on \( F^k(M) \).
Proof. Since the fixed points are of order \( k_0 \), the action of \( G \times M \) on \( F^k(M) \) is locally free for \( k \geq k_0 + 1 \). Because the vector fields \( F^k(\xi_x) \) on \( F^k(M) \) are nonvanishing, \( \mathcal{L} \) is a sheaf and \( L^k \) is a smooth subbundle of \( J^k(TM) \). Since \( g_{\xi_x}^* \) is the kernel of \( L^{r+1} \rightarrow L^r \), it is zero for \( r \geq k \). It follows that \( \mathcal{L} \) is a LAS of order \( k \). The groupoid integrating the Lie algebroid \( L^k \) is the action groupoid with the canonical action on \( F^k(M) \), so Theorem 1.17 implies that \( \mathcal{L} \) is the sheaf of vector fields \( v \) such that for each \( k \)-frame \( f_x \), there exists an \( X \in \mathfrak{g} \) such that \( F^k(v)|_{f_x} = F^k(\xi_x)|_{f_x} \). \( \square \)

1.5 Transitive sheaves

The situation becomes especially transparent if the sheaves of Lie algebras are transitive.

Definition 1.20. Let \( \mathcal{L} \) be a sheaf of Lie algebras of vector fields on a connected manifold \( M \). Then \( \mathcal{L} \) is called a transitive Lie Algebra Sheaf if \( L^0_x = T_x M \) for all \( x \in M \), if the pushforward \( \exp(tX)_x : \mathcal{L}_x \rightarrow \mathcal{L}_y \) yields an isomorphism \( L_x \rightarrow L_y \) for all \( x, y \in M \) with \( y = \exp(tX)(x) \) and if \( \mathcal{L} \) is determined by \( L \) in the sense that \( j^\infty(v) \in L_x \) for all \( x \in U \) implies \( v \in \mathcal{L}(U) \).

This coincides with the notion of a transitive LAS in the sense of Singer-Sternberg, cf. [3], def. 1.3, 1.4 and 1.8. We set out to prove that every transitive LAS is a LAS of finite order in the sense defined before.

Proposition 1.21. For every transitive LAS, the subset \( L \subseteq J^\infty \) is a smooth locally trivial bundle of Fréchet Lie algebras over \( M \). The same holds for all bundles \( L^k \rightarrow M \).

Proof. Since \( \mathcal{L} \) is transitive, every \( x_0 \in M \) possesses a neighbourhood \( U \) with a local frame \( X_1, \ldots, X_d \) of sections of \( \mathcal{L}(U) \). For \( \bar{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \) sufficiently close to zero, \( \phi_{\bar{x}} = \exp(x_1 X_1) \circ \cdots \circ \exp(x_n X_n) \) is well defined. By shrinking \( U \) if necessary, we obtain a chart \( \kappa^{-1} : \mathbb{R}^d \supseteq V \rightarrow U \subseteq M \) by \( \kappa^{-1}(\bar{x}) = \phi_{\bar{x}}(x_0) \). The map \( V \times L_{x_0} \rightarrow L \) defined by \( (\bar{x}, Y) \mapsto \phi_{\bar{x}} Y \) is a local trivialisation of \( L \) over \( U \) and every two such trivialisations differ by a smooth isomorphism of the trivial bundles \( V \times \kappa \kappa^{-1}(V') \times L_{x_0} \rightarrow V' \times \kappa \kappa^{-1}(V) \times L_{x_0} \). Since these isomorphisms preserve the filtration, not only \( L \rightarrow M \) but also all the \( L^k \rightarrow M \) are smooth. \( \square \)

By Theorem 1.5, transitivity of \( L_x \), i.e. the requirement \( \mathfrak{g}_x^{-1} = T_x M \) for all \( x \in M \), implies that \( L_x \) has finite order. Because all \( L_x \) are isomorphic, the order is locally constant, hence finite. We arrive at the following proposition.

Proposition 1.22. Every transitive LAS is a LAS of finite order that is regular of order 0. For all \( k \in \mathbb{N} \), the bundle \( L^k \rightarrow M \) is a transitive Lie algebroid.

We can now apply Theorem 1.17 to obtain a transitive groupoid \( G^L \), the source fibre at of which is a principal fibre bundle over \( M \).

Theorem 1.23 (Cartan I for Transitive LAS). If \( \mathcal{L} \) is a transitive LAS, then there exists an immersed principal subbundle \( P_L \subseteq F^k(M) \) whose structure group has Lie algebra \( L_{x_0} / L_{x,k+1} \), such that \( \mathcal{L} \) is the sheaf of infinitesimal symmetries of \( P_L \), in the sense that \( v \in \text{Vec}(U) \) is in \( \mathcal{L}(U) \) if and only if \( F^k(v)|_p \) is in \( T_p P_L \) for all \( p \in P_L \).
In other words: every transitive LAS is the sheaf of symmetries of an im-
mersed subbundle of the $k$th order frame bundle.

Proof. Since $L^k$ is transitive, so is $G^k_L := M$, and its source fibre at $x_0 \in M$ is
the principal fibre bundle $G^k_{L,x_0} \to M$. The Lie algebra of its structure group
$G^k_{L,x_0}$ is the kernel of the anchor $L^k_{x_0} \to L^0_{x_0}$. Since $\iota: G^k_L \to J^k(M,M)$ is an
immersion, the kernel $K$ of the map $\iota: G^k_{L,x_0} \to G^k_{L,x_0}$ is a closed discrete normal subgroup of $G_{L,x_0,x_0}$. Now $P := G^k_{L,x_0}/K$ is a principal fibre bundle with
structure group $G^k_{L,x_0,x_0}/K$. If we identify the source fibre at $x_0$ of $J^k(M,M)$
with $F^k(M)$, then the equivariant immersion $P \to F^k(M)$ is injective, and the
distribution $\Delta^k$ on $F^k(M)$ consists of the translates of $TP \subseteq TF^k(M)$ by the
structure group $Gl^k(n)$.

Let us consider the images $P^j_L$ of $P_L$ under the projection maps $\pi^j_L: F^j(M) \to
F^j(M)$. For each $j = 0, \ldots, k$, we have an inclusion $\iota_j: P^j_L \hookrightarrow F^j(M)$. The im-
mersed subbundle $P^1_L$ of $F^1(M)$ is a subbundle of the ‘ordinary’ frame bundle,
hence a $G$-structure on $M$. Its structure Lie algebra is $g_{x_0}^0$.

We can consider $P^j_L \subseteq F^2(M)$ as a principal fibre bundle over $P^1_L$ with
structure Lie algebra $g^1_{x_0}$, but also as a subbundle of the prolongation $(P^j_L)^{(1)} \to
P^1_L$ with structure Lie algebra $g^1 \subseteq g^{(1)}$. Continuing in this way, we obtain a
tower of principal fibre bundles

$$P^k_L \to \cdots \to P^1_L \to M$$

such that $P^k_L = P_L$ and $P^{j+1}_L \to P^j_L$ has structure Lie algebra $g^j_{x_0}$. This can be
regarded as a higher order $G$-structure.

References


