1 Truncated Lie algebras

Fifth lecture on Singer and Sternberg’s 1965 paper [1], by Bas Janssens.

1.1 Introduction

Let $V$ be a finite dimensional vector space. On the bosonic Fock space $S(V)$, we define the creation operator for $v \in V$ by $a_B^*(v) : u \mapsto v \vee u$ and the annihilation operator for $\alpha \in V^*$ by $a_B(\alpha) : u \mapsto i_\alpha u$ (contraction with $\alpha$). Similarly, we define creation and annihilation operators on the fermionic Fock space $V$. Its homology is called the Spencer homology which is a finitely generated $V$-module. We have seen before that this is equivalent to $S(V)^* \otimes \Lambda(V)$, hence it is a finitely generated $V^*$-module. Since $\text{ann}(\mathfrak{g}) = S(V)^* \otimes W^* / \text{ann}(\mathfrak{g})$ is a finitely generated $S(V)$-module. Since $[\partial^*, a_B^*(v)] = 0$, the kernel $\text{Ker}(\partial^*) \subseteq \mathfrak{g}^* \otimes \Lambda V$ is an $S(V)$-submodule of a finitely generated $S(V)$-module, hence finitely generated itself by the Hilbert Basis Theorem. But $a_B^*(V) \text{Ker}(\partial^*) \subseteq \text{Im}(\partial^*)$ because $\{\partial^*, a_B^*(v)\} = a_B^*(v)$, so $\text{Ker}(\partial^*) / \text{Im}(\partial^*)$ is not only finitely generated, but even finite dimensional.

\[
\partial^* = \sum_i a_B^*(e_i) \otimes a_F(\epsilon^i)
\]

where $(e_1, \ldots, e_n)$ is a basis of $V$ and $(\epsilon^1, \ldots, \epsilon^n)$ the dual basis of $V^*$. Explicitly, it satisfies

\[
\partial^*(u_1 \vee \ldots \vee u_k) \otimes (v_1 \wedge \ldots \wedge v_l) = \sum_{j=1}^l (-1)^j (v_j \vee u_1 \vee \ldots \vee u_k) \otimes (v_1 \wedge \ldots \hat{v}_j \ldots \wedge v_l).
\]

Using the $[a_B(\alpha), a_B^*(v)] = \alpha(v) 1$ (CCR) and $\{a_F(\alpha), a_B^*(v)\} = \alpha(v) 1$ (CAR), one verifies the following commutation relations.

**Proposition 1.1.** We have $[\partial^*, a_B^*(v)] = 0$ and $\{\partial^*, a_F^*(v)\} = a_B^*(v)$.

Let $W$ be a finite dimensional vector space. We continue to denote by $\partial^*$ the extension of $\partial^*$ to $S(V) \otimes W^* \otimes \Lambda V$ by $\partial^*$ and we denote its dual on $\mathfrak{g}^* \otimes W \otimes \Lambda V^*$ by $\partial$. Let $\mathfrak{g} \subseteq \mathfrak{g}^* \otimes W \otimes \Lambda V^*$ be a graded subspace such that $\text{ann}(\mathfrak{g}) \subseteq S(V) \otimes W^*$ is a $S(V)$-module. We have seen before that this is equivalent to $\mathfrak{g} \subseteq \text{pro}(\mathfrak{g})$ for all $k$. In this case, the operator $\partial$ restricts to $\mathfrak{g} \otimes \Lambda V \subseteq S(V^*) \otimes W \otimes \Lambda V$. Its homology is called the Spencer homology of $\mathfrak{g}$.

**Theorem 1.2.** The Spencer homology is finite dimensional.

**Proof.** The Spencer homology is dual to the cohomology of the operator $\partial^*$ on $\mathfrak{g}^* \otimes \Lambda V$. Since $\text{ann}(\mathfrak{g})$ is an $S(V)$-module, the quotient $\mathfrak{g}^* = S(V)^* \otimes W^* / \text{ann}(\mathfrak{g})$ is a finitely generated $S(V)$-module. Since $[\partial^*, a_B^*(v)] = 0$, the kernel $\text{Ker}(\partial^*) \subseteq \mathfrak{g}^* \otimes \Lambda V$ is an $S(V)$-submodule of a finitely generated $S(V)$-module, hence finitely generated itself by the Hilbert Basis Theorem. But $a_B^*(V) \text{Ker}(\partial^*) \subseteq \text{Im}(\partial^*)$ because $\{\partial^*, a_B^*(v)\} = a_B^*(v)$, so $\text{Ker}(\partial^*) / \text{Im}(\partial^*)$ is not only finitely generated, but even finite dimensional.
References