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Mini-Workshop: Reflection Positivity in Representation Theory, Stochastics and Physics

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Abstract. The central focus of the workshop was reflection positivity, its occurrence in physics, representation theory, abstract harmonic analysis, and stochastic analysis. The program was intrinsically interdisciplinary and included talks covering different aspects of reflection positivity.

Mathematics Subject Classification (2010): 22E70, 17B10, 22E65, 81T08.

Introduction by the Organisers

The mini-workshop on *Reflection Positivity in Representation Theory, Stochastics and Physics* organized by P. Jorgensen (University of Iowa), K-H. Neeb (University of Erlangen-Nürnberg), and G. Ólafsson (Louisiana State University) was held during the week Nov. 30 to Dec. 6, 2014. It was organized around the concept of *reflection positivity*, a central theme at the crossroads of the theory of representations of Lie groups, harmonic analysis, stochastic processes, and constructive quantum field theory. It employs tools and ideas from different branches of mathematics. The workshop consisted of seventeen scientific presentations and four problem sessions focused around nineteen problems from mathematics and physics (see separate abstract).

Reflection positivity is one of the axioms of *constructive quantum field theory* as they were formulated by Osterwalder and Schrader 1973/1975. In short, the goal is to build a bridge from a euclidean quantum field to a relativistic quantum field by analytic continuation to imaginary time. In terms of representation theory this can be formulated as transferring representations of the euclidean motion group to a unitary representation of the Poincaré group via c-duality of symmetric pairs. This duality is defined as follows. If $\mathfrak{g}$ is a real Lie algebra and $\tau : \mathfrak{g} \to \mathfrak{g}$ an involution, then we write $\mathfrak{h} = \mathfrak{g}^\tau$ and $\mathfrak{q} = \mathfrak{g}^{-\tau}$ for the τ-eigenspaces and observes that $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$ inherits a natural Lie algebra structure and an involution $\tau^c(x + iy) = x - iy$. We write G and G^c for corresponding Lie groups. The representation theoretical task is to start with a representation $(\pi, \mathcal{E})$ of G and, via the derived representation of $\mathfrak{g}$ and Osterwalder-Schrader quantization, transfer it to a unitary representation $(\pi^c, \widehat{\mathcal{E}})$ of G^c. Several of the participants have worked on developing the basic ideas of this transfer process as well as applying it to concrete groups and representations.

The motivation for duality and reflection positivity in Quantum Field Theory is different from that from representations of Lie groups. Constructive quantum field theory has its origin in Wightman's axioms, and in the work of Osterwalder–Schrader based on reflection positivity, and euclidean invariance. In the original variant, based on Wightman, the quantum fields are operator valued distributions, but the subsequent heavy task of constructing models with non-trivial interaction has proved elusive. New approaches emerged, e.g., renormalization, and analytic continuation to euclidean fields. One reason for the latter is that, in the euclidean approach, the operator valued distributions (unbounded operators, and non-commutativity), are replaced by (commuting) systems of reflection positive stochastic processes, i.e., systems of random variables satisfying reflection symmetry as dictated by the axioms of Osterwalder–Schrader.

Finally, the last decade witnessed an explosion in new research directions involving stochastic processes, and neighboring fields. This has entailed an expansion of the more traditional tools based on Ito calculus; expanding to such infinite-dimensional stochastic calculus models as Malliavin calculus, but also research by some of the proposed workshop participants, for example, processes whose square-increments are stationary in a generalized sense, and associated Gaussian, and non-Gaussian, processes governed by singular measures, and constructed with the use of renormalization techniques.

One of the goals of the workshop was to build bridges between these directions and initiate discussions and an exchange of ideas between researchers in those different fields. This culminated in the nineteen open problems that were proposed and discussed by the participants. We refer to the abstracts for a list of problems and some explanations.

The talks included several aspects of reflection positivity, such as:

(1) Basic mathematical background for reflection positive Hilbert spaces, Osterwalder-Schrader quantization and reflection positive representations.
(2) Reflection positive representations and their integration
(3) Basic introduction to reflection positivity
(4) Reflection positivity in stochastic analysis and spectral theory
(5) Stochastic quantization
(6) The connection between complex measures, positivity and reflection positivity

(7) Analysis on path groups, gauge groups, and some other infinite dimensional groups
(8) Lie supergroups

Acknowledgments. The organizers thank the director Prof. Dr. Gerhard Huisken, and the Oberwolfach staff for offering us outstanding support in all phases of the planning.

Mini-Workshop: Reflection Positivity in Representation Theory, Stochastics and Physics

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Abstracts

I: Harmonic Analysis on Lie Supergroups — An Overview
II: Riesz Superdistributions

Alexander Alldridge

Informally, a *supermanifold* X is a space with local coordinates u, ξ such that:

$$u^a u^b = u^b u^a, \quad u^a \xi^i = \xi^i u^a, \quad \xi^i \xi^j = -\xi^j \xi^i$$

in the $\mathbb{K}$-algebra of functions. Formally, it has a Hausdorff underlying topological space and is locally isomorphic—in the ambient category of locally super-ringed spaces with $\mathbb{K}$-algebra sheaves of functions—to the model space $(\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p} \otimes \bigwedge(\mathbb{K}^q)^*)$.

Following the point of view introduced by Berezin and Kac, a *Lie supergroup* is a group object in this category. As observed by Kostant and Koszul, this is equivalent to the more tangible data of *supergroup pairs* $(\mathfrak{g}, G_0)$ where G_0 is a real Lie group and $\mathfrak{g}$ is a $\mathbb{K}$-Lie superalgebra such that $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G_0) \otimes_{\mathbb{R}} \mathbb{K}$, carrying a linear G_0-action Ad by Lie superalgebra automorphisms which extends the adjoint action of G_0 and whose differential $d\mathrm{Ad}$ is the restriction of the bracket of $\mathfrak{g}$.

By way of an example, we construct the *meta-spin supergroup* by its associated supergroup pair $(\mathfrak{s}, \tilde{S}_0)$: Fix a supersymplectic super-vector space (W, ω) of finite even dimension over $\mathbb{C}$. The *Clifford–Weyl algebra* $\mathcal{CW}$, is the unital enveloping superalgebra of the Heisenberg–Clifford superalgebra $\mathfrak{h} = W \times \mathbb{C}$ with the relations $[w, w'] = \omega(w, w')$. Its filtred part $\mathcal{CW}_{\leqslant 2}$ is a Lie superalgebra of the form $\mathfrak{s} \ltimes \mathfrak{h}$ where $\mathfrak{s} = \mathfrak{spo}(W, \omega)$ is the symplectic-orthogonal Lie superalgebra of (W, ω).

Any polarisation $W = V_+ \oplus V_-$ turns $\mathbb{C}[V_+] = S(V_-)$ into a $\mathcal{CW}$-module, the *oscillator-spinor module*, by letting V_- act by multiplication and V_+ by derivation. Any choice of Hermitian inner product $\langle \cdot \mid \cdot \rangle$ on V_+ for which the graded parts are orthogonal defines an antilinear even isomorphism $V_+ \longrightarrow V_-$. The fixed set of its extension to an even antilinear involution of W is denoted by $W_{\mathbb{R}}$. Then $\mathfrak{s}_{\mathbb{R}} := \underline{\mathrm{End}}(W_{\mathbb{R}}) \cap \mathfrak{s}$ is the purely even Lie superalgebra $\mathfrak{sp}(W_{\mathbb{R},\bar{0}}, \Im\langle \cdot \mid \cdot \rangle) \times \mathfrak{o}(W_{\mathbb{R},\bar{1}}, \Re\langle \cdot \mid \cdot \rangle)$. Taking S_0 to be $\mathrm{Sp} \times \mathrm{SO}$, we obtain a supergroup pair $(\mathfrak{s}, S_0)$. A double cover $(\mathfrak{s}, \tilde{S}_0)$ is given by replacing S_0 by $\mathrm{Mp} \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}$.

A linear action of a Lie supergroup G on a finite-dimensional super-vector space V is the same thing as a pair $(d\pi, \pi_0)$ of actions of the associated supergroup pair $(\mathfrak{g}, G_0)$, where $d\pi$ extends the differential of π_0 and is G_0-equivariant for the adjoint action. These also makes sense in infinite dimensions. There is an equivalent characterisation as representations of suitable convolution algebras, see Ref. [1]. For G_0 reductive, Schwartz class versions thereof lead to moderate growth representations and the generalisation of the Casselman–Wallach globalisation theorem.

The second part of the first talk focused on unitary representations of a Lie supergroup G with associated supergroup pair $(\mathfrak{g}, G_0)$. We suggested to call a continuous representation $(d\pi, \pi_0)$ on a graded Hilbert space $\mathcal{E}$ *weakly unitary* if π_0 is unitary and $\mathcal{E}_\infty$ is contained in the domain of $d\pi(x)^*$ for any $x \in \mathfrak{g}_{\bar{1}}$. For

$\mathbb{K}=\mathbb{R}$, the stronger notion of *unitarity* has been advocated by by Varadarajan et al. Here, one requires $e^{i\pi/4}d\pi(x)$ to be skew-adjoint for any $x\in\mathfrak{g}_{\bar{1}}$.

Unitary representations are special unless $\mathfrak{g}_{\bar{1}}$ is 'small': If $\mathfrak{g}_{\bar{0}}=[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}]$ and $\mathcal{E}$ is an irreducible unitary representation $\mathcal{E}$, then $\mathcal{E}$ is a generalised highest weight module [9]. On the other hand, if $G_0=(\mathbb{R},+)$ and $\mathfrak{g}$ is the Clifford Lie superalgebra of dimension $n|1$, the abstract Fourier transform for weakly unitary representations admits Fourier inversion formula and a Paley–Wiener theorem for the Schwartz class [2]. Thus, for the purposes of Harmonic Analysis, weakly unitary representations might be more useful.

An example of a weakly unitary representation of the meta-spin supergroup pair $(\mathfrak{s},\tilde{S}_0)$ is given by introducing a pre-Hilbert structure on the oscillator-spinor module: Elements of $\mathbb{C}[V_+]$ give superfunctions on $V_{\bar{0}}\times V_{\bar{1}}\otimes_{\mathbb{R}}\mathbb{C}$. Here, we have a natural conjugation, and may define $(f_1|f_2):=\int_{V_{\bar{0}}\times V_{\bar{1}}\otimes_{\mathbb{R}}\mathbb{C}}|D(z,\bar{z})|\,e^{-z\bar{z}}\bar{f}_1 f_2$.

The completion $\mathcal{E}$ is $\mathcal{E}_0\otimes\bigwedge V_{-,\bar{1}}$ where $\mathcal{E}_0$ is the classical Bargmann–Fock space. By results of Howe [6] (see also Ref. [7]), the representation integrates to a unitary representation of $\tilde{S}_0$. Moreover, if $\tilde{K}_0\subseteq\tilde{S}_0$ is a maximal compact subgroup, then the space of $\tilde{K}_0$-finite vectors $\mathbb{C}[V_+]$ is a Harish-Chandra $(\mathfrak{s},\tilde{K}_0)$-module. Thus, the $\mathfrak{g}$-action extends to $\mathcal{E}_\infty$, and $\mathcal{E}$ is weakly unitary. (See also Ref. [10].)

We concluded the first talk with an application to a problem from Random Matrices taken from Refs. [4, 7], which we believe is connected to Reflection Positivity (or RP). Let G' be one of $\mathrm{U}(N)$, $\mathrm{USp}(N)$, $\mathrm{O}(N)$, or $\mathrm{SO}(N)$, where N is even in the second case and $\mathrm{USp}(N)$ denotes the quaternionic unitary group. Let $\mathfrak{g}'$ be the Lie algebra of G' and $\mathfrak{g}$ its commutant in $\mathfrak{s}=\mathfrak{spo}(W,\omega)$ where $W=V_+\oplus V_+^*$ with its standard symplectic form. Here, $V_+=U\otimes\mathbb{C}^N$ where $\dim U=n|n$. Then $\mathfrak{g}$ is $\mathfrak{gl}(U)$, $\mathfrak{spo}(U\oplus U^*)$, respectively, in the first two cases, and $\mathfrak{osp}(U\oplus U^*)$ in the latter two. Let G_0 be of type $\mathrm{U}\times\mathrm{U}$, $\mathrm{SO}^*\times\mathrm{USp}$, or $\mathrm{Mp}\times_{\mathbb{Z}/2\mathbb{Z}}\mathrm{Spin}$, respectively.

Let $\mathcal{E}$ be the oscillator-spinor representation of $(\mathfrak{s},\tilde{S}_0)$. Then the representation of $(\mathfrak{g},G_0)$ on $\mathcal{E}^{G'}$, the space of G'-fixed vectors, is irreducible, weakly unitary, and extends analytically to the supersemigroup pair $\Gamma=(\mathfrak{g},\Gamma_{\bar{0}}\times G_{\bar{1},\mathbb{C}})$ where $\Gamma_{\bar{0}}$ is a compression semigroup or a double cover thereof and $G_{\bar{1},\mathbb{C}}$ is the complex Lie group $\mathrm{GL}(U_{\bar{1}})$, $\mathrm{Sp}(U_{\bar{1}}\oplus U_{\bar{1}})$, or $\mathrm{Spin}_{\mathbb{C}}(U_{\bar{1}}\oplus U_{\bar{1}}^*)$, respectively. The character χ of this representation is an analytic superfunction on Γ. For any $t=(e^{\phi_1},\dots,e^{\phi_n},e^{i\psi_1},\dots,e^{i\psi_n})$, $\Re\phi_j>0$, we have for connected G':

$$\int_{G'}\prod_{j=1}^{n}\frac{\det_N(e^{i\psi_j/2}-e^{-i\phi_j/2}k)}{\det_N(e^{\psi_j/2}-e^{-\phi_j/2}k)}\,dk=\chi(t),$$

and there is a Weyl-type formula. A similar statement holds for $G'=\mathrm{O}(N)$.

We now come to the second talk, devoted to the Superbosonisation Identity of Littelmann–Sommers–Zirnbauer [8]. The motivating problem is the asymptotic expansion, as $N\longrightarrow\infty$, of the averaged Green's function of an ensemble of Hermitian matrices with $\mathrm{U}(N)$-invariant law $\mu\equiv\mu_N$. It is a derivative of the partition function $Z(\alpha,\beta):=\int_{i\mathfrak{u}(N)}\det\big(\frac{H-\beta}{H-\alpha}\big)d\mu(H)$, which, by introducing fermionic 'ghosts', may be rewritten as a Berezin integral. The Superbosonisation Identity converts this into an expression amenable to asymptotic expansion.

Consider $U = \mathbb{C}^{p|q}$, $V = U \otimes \mathbb{C}^N$, and $W = V \oplus V^*$. Let f be a holomorphic superfunction on the superdomain in $\mathbb{C}^{p|q\times p|q}$ over $T_{\bar{0}} \times \mathbb{C}^{q\times q}$, where $T_{\bar{0}}$ is the tube over $\mathrm{Herm}^+(p)$. Assume that f and its derivatives satisfy Paley–Wiener estimates along $\mathrm{Herm}^+(p)$, locally uniformly in $\mathbb{C}^{q\times q}$. Then

$$\int_{V\oplus V^*} |D(\psi,\bar{\psi})|\, f(\psi\bar{\psi}) = C_N \int_\Omega |Dy|\, \mathrm{Ber}(y)^N f(y),$$

where Ω is a homogeneous totally real subsupermanifold of $\mathbb{C}^{p|q\times p|q}$ with underlying manifold $\Omega_0 = \mathrm{Herm}^+(p) \times \mathrm{U}(q)$ and C_N is a constant. As observed in Ref. [3], its precise value is $C_N = \sqrt{\pi}^{Np}\, \Gamma_\Omega(N,\dots,N)^{-1}$, where the gamma function $\Gamma_\Omega(\mathbf{m}) := \int_\Omega |Dy| e^{-\mathrm{str}(y)} \Delta_{\mathbf{m}}(y)$, for $\Delta_{\mathbf{m}}(Z) := \prod_{k=1}^{p+q} \mathrm{Ber}((Z_{ij})_{i,j\leqslant k})^{m_k - m_{k+1}}$ and $\mathbf{m} = (m_1,\dots,m_{p+q})$.

This results from the interpretation of the right-hand side of the Superbosonisation Identity as a functional generalising the classical Riesz distribution on $\mathrm{Herm}^+(p)$ and the Cauchy integral on $\mathrm{U}(q)$. By use of Harmonic Analysis, we find

$$\Gamma_\Omega(\mathbf{m}) = (2\pi)^{p(p-1)/2} \prod_{j=1}^p \Gamma(m_j - j + 1) \prod_{k=1}^q \frac{\Gamma(q-k+1)}{\Gamma(m_{p+k}+q-k+1)} \frac{\Gamma(m_{p+k}+k)}{\Gamma(m_{p+k}-p+k)}.$$

Classically, the Riesz distributions give rise to the unitary structure on limits of holomorphic discrete series representations of $\mathrm{SU}(p,p)$. The poles of the gamma function of $\mathrm{Herm}^+(p)$ are closely related to the discrete points of the Wallach set. Recent work of Neeb and Ólafsson relates this circle of ideas to RP.

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Energy and Brownian representations of path groups

Maria Gordina
(joint work with A. M. Vershik, B. Driver, S. Albeverio)

This is a report on a joint work with A. M. Vershik, B. Driver, S. Albeverio. We study two unitary representations of (infinite-dimensional) groups of paths with values in a compact Lie group. Before stating the results and open problems, we can can refer to [6, p. 263] which explains how these representations are connected to the QFT and reflection positivity.

Now we proceed to the main object of this project. Let G be a compact connected Lie group, and let $\mathfrak{g}$ be its Lie algebra. We consider the group of finite energy paths in G, namely,

$$H(G) = \Big\{ h : [0,1] \to G, h(0) = e, \ h \text{ is absolutely continuous such that } \|h\|_H^2 := \int_0^1 |h^{-1}(s)\, h'(s)|_{\mathfrak{g}}^2 ds < \infty \Big\}.$$

This is a group with respect to pointwise multiplication and topology determined by the norm $\|\cdot\|_H$. There are two unitary representations of this group that were previously studied, some of the references include [3, 4].

The first one is in $L^2(W(G), \mu)$, where $W(G)$ is the Wiener space of continuous paths in G, and μ is the Wiener (non-Gaussian) measure. Then the Brownian representation is a unitary representation induced by the action of $H(G)$ on $W(G)$ by left or right multiplication, L_φ and R_φ. The probability measure μ is known to be quasi-invariant under these actions, and therefore we can define the **right Brownian measure representation** U^R by

$$\left(U_\varphi^R f\right)(g) := \left(U^R(\varphi) f\right)(g) = \left(\frac{d\left(\mu \circ R_\varphi^{-1}\right)}{d\mu}(g)\right)^{1/2} f(g\varphi)$$

and the **left Brownian measure representation** U^L by

$$\left(U_\varphi^L f\right)(g) := \left(U^L(\varphi) f\right)(g) = \left(\frac{d\left(\mu \circ L_\varphi^{-1}\right)}{d\mu}(g)\right)^{1/2} f\left(\varphi^{-1} g\right)$$

for any $f \in L^2(W(G), \mu)$, $\varphi \in H(G)$, $g \in W(G)$.

The **energy representation** is a unitary representation of $H(G)$ on the space $L^2(W(\mathfrak{g}), \nu)$, where ν is the Gaussian measure. Let

$$\left(E_\varphi f\right)(w.) := e^{i\int_0^{\cdot}\langle\varphi^{-1}d\varphi, dw_s\rangle} f\left(\int_0^{\cdot} \mathrm{Ad}_{\varphi^{-1}}\, dw_s\right).$$

for any $\varphi \in H(G)$, $f \in L^2(W(\mathfrak{g}), \nu)$. E_φ is called the **energy representation** of $H(G)$.

While by now we know some of the properties of these representations such as cyclicity and unitarily equivalence as stated in the theorem below, some questions are still open.

Theorem 1. (1) Cyclicity of **1**: *the space*

$$\mathrm{Span}\left\{ h_\varphi(g_t) = \left(\frac{d\left(\mu \circ R_\varphi^{-1}\right)}{d\mu}(g)\right)^{1/2},\ \varphi \in H(G) \right\}$$

is dense in $L^2(W(G), \mu)$.

(2) *Both* U^R *and* U^L *are unitarily equivalent to the energy representation* E.

The open questions for which we have partial answers to are the following:

(1) whether the von Neumann algebras generated by the left and right Brownian representations are commutants of each other. This is a well-known fact in the locally compact case, but clearly these techniques are not applicable for the group $H(G)$;
(2) whether the left and right Brownian representations are factorial. The main ingredient in the proof published in [2] seems to be wrong, and therefore we need a very different approach;
(3) if it is the case, a natural question is to find the type of this factor.

Our main tool in attacking these questions are techniques developed by M. Rieffel *et al* for example in [5]. Moreover, this might provide a very concrete example where the modular theory of Tomita-Takesaki is applicable.

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Introduction to Reflection Positivity

Arthur Jaffe

The condition of reflection positivity (RP) was discovered in 1972 by Konrad Osterwalder and Robert Schrader, while they were both working at Harvard University with me as post-doctoral fellows [8, 9]. This discovery solved a major problem at the time: if one could construct a Euclidean field theory, how could one use that data to recover a relativistic quantum field?

In fact RP turned out to be the central feature that related probability of classical fields to quantum theory. It gave a Hilbert space inner-product (that turned out to be a *quantization algorithm* for states of a classical field). It also provided a quantization algorithm for operators: the quantization of time translation yields a self-adjoint, contraction semi-group whose generator is the positive Hamiltonian of quantum theory. The quantization of the classical field is the quantum field, continued to imaginary time. And the quantization of the full Euclidean group, yields an analytic continuation of a unitary representation of the Poincaré group.

Reflection positivity also provided a useful tool in analyzing phase transitions for the classical field. The new RP inner product provided reflection and infra-red bounds that were crucial in establishing the existence of phase transitions. For the associated quantum fields, these phase transitions were reflected with symmetry breaking and non-unique ground states.

RP also turned out to be key in analyzing lattice statistical mechanical models, in establishing the existence of phase transitions, and in studying the degenerate ground states of certain modes that occur in quantum information theory.

RP also provides a tool to analyze fermions and gauge theories, as well as complex functionals for classical fields. These do not necessarily have a positive functional integral. But the RP-functional can play an important substitute role, yielding multiple reflection bounds and their consequences.

A number of early references can be found in [1]. Recent work on quantum information with Majoranas [2, 3, 4] and with Para-Fermions [5] were covered in the talk. The work on complex functionals appears in [6, 7]. Other related work is in progress.

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Stochastic Quantization

Arthur Jaffe

We compare the standard method of quantization through functional integration satisfying reflection positivity, see for example [1] and the references in [2] with the method of stochastic quantization, introduced by Nelson in 1966 [5] and extended by Parisi and Wu [5, 6]. Stochastic quantization has not yet been effectively implemented to give examples of non-linear quantum field theories, with a Hilbert space of states, and the other usual features of quantum theory. Recently Hairer investigated the method and made progress in classifying the renormalization and solutions to the classical stochastic partial differential equations involved [4].

Here we study the stochastic quantization of a free field. This leads to a linear PDE with the *stochastic time* λ, namely

$$\frac{\partial \Phi_\lambda(x)}{\partial \lambda} = -\frac{1}{2}\left(-\Delta + m^2\right)\, \Phi_\lambda(x) + \xi_\lambda(x)\ .$$

Here $\xi_\lambda(x)$ is a forcing term. One can solve the equation for given initial data at $\lambda = 0$ and given $\xi_\lambda(x)$. One then considers a white-noise distribution of the forcing term $\xi_\lambda(x)$. For simplicity we consider here vanishing initial data, $\Phi_0(x) = 0$.

It is elementary to see that the Gaussian white noise leads to a probability distribution $d\mu_\lambda(\Phi)$ for the solution to this equation at stochastic time λ. This measure is defined by its moments. We have proved

Theorem. [3] *For stochastic time $\lambda < \infty$, the measure $d\mu_\lambda(\Phi)$ does not satisfy reflection positivity with respect to reflection ϑ of the physical time, $\vartheta : t \mapsto -t$.*

This result extends to non-Gaussian quantum measures for small perturbations of a Gaussian, in case the perturbation is continuous. As a consequence of this theorem, one has to question how to modify stochastic quantization, in order to obtain a quantum field theory with the usual quantum mechanics and Hilbert space.

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Positive energy representations of gauge groups

Bas Janssens
(joint work with Karl-Hermann Neeb)

This is an extended abstract of a presentation held at the Oberwolfach Seminar 'Reflection Positivity in Representation Theory, Stochastics and Physics' on December 4, 2014.

1. Definitions

In order to define *gauge groups*, we need 3 ingredients: a smooth manifold M, a compact simple Lie group K, and a principal K-bundle $\pi\colon P \to M$. The gauge group $\mathrm{Gau}_c(P)$ is then the group of compactly supported ('pure gauge') vertical automorphisms of P. The *gauge algebra* $\mathfrak{gau}_c(P)$ is the Lie algebra of compactly supported smooth sections of the adjoint bundle $\mathrm{Ad}(P) := P \times_{\mathrm{Ad}} \mathfrak{k}$.

Energy is conjugate to time, so in order to define positive energy representations, we need one extra ingredient: a *time translation.* This is a smooth action $T\colon \mathbb{R} \to \mathrm{Aut}(P)$ of $\mathbb{R}$ on P by bundle automorphisms, with the property that the induced flow $T_M\colon \mathbb{R} \to \mathrm{Diff}(M)$ is locally free. Both the time flow $v := \frac{d}{dt}|_0 T(t)$ in $\mathrm{Vec}(P)$ and its projection to M, $v_M := \frac{d}{dt}|_0 T_M(t) \in \mathrm{Vec}(M)$, are everywhere nonzero. Using this extra datum T, we construct the extended gauge group $G := \mathrm{Gau}_c(P) \rtimes_T \mathbb{R}$ and Lie algebra $\mathfrak{g} := \mathfrak{gau}_c(P) \rtimes_v \mathbb{R}$, where the twist in the semidirect product comes from considering an element $T(t) \in \mathrm{Aut}(P)$ as an automorphism of $\mathrm{Gau}_c(P)$.

A *unitary representation* of a locally convex Lie group G is a group homomorphism $\rho\colon G \to \mathrm{U}(\mathcal{H})$ into the group of unitary operators on a Hilbert space. However, because the phase factor is irrelevant, we are interested in *projective* unitary representations of G, that is, group homomorphisms $\overline{\rho}\colon G \to \mathbb{P}\mathrm{U}(\mathcal{H})$ into the *projective* unitary group. In joint work with K.-H. Neeb, we have classified, in the context of compact space-time manifolds M and periodic time translations T, the projective unitary representations of G which are *smooth* and of *positive energy* [2].

A ray $[\psi] \in \mathbb{P}(\mathcal{H})$ is called *smooth* if the orbit map $G \to \mathbb{P}(\mathcal{H})\colon g \mapsto \overline{\rho}(g)[\psi]$ is smooth, and a projective representation $\overline{\rho}$ is called *smooth* if the set $\mathbb{P}(\mathcal{H})^\infty$ of smooth rays is dense in $\mathbb{P}(\mathcal{H})$. The requirement that a projective unitary representation be smooth is very natural in the context of Lie theory for infinite dimensional groups, as it ensures the existence of a common domain $\mathcal{H}^\infty$ for the derived projective Lie algebra representation $\mathtt{d}\overline{\rho}\colon \mathfrak{g} \to \mathbb{P}\mathrm{End}(\mathcal{H}^\infty) = \mathrm{End}(\mathcal{H}^\infty)/\mathbb{C}\mathrm{id}$. Smoothness should probably not be seen as a serious restriction on our result.

More serious, but still not overly restrictive, is the requirement that $\overline{\rho}$ be of *positive energy.* This is the requirement that the *Hamilton operator* $H := -i\mathtt{d}\overline{\rho}(0, 1)$, the selfadjoint generator of time translations (well defined up to scalars), have a lower bound on its spectrum.

2. Special cases

An important special case of this construction is when $P = M \times K$ is trivial (and trivialised), and T is the lift to P of a locally free $\mathbb{R}$-action T_M on M. We then have $G = C_c^\infty(M,K) \rtimes_{T_M} \mathbb{R}$ and $\mathfrak{g} = C_c^\infty(M,\mathfrak{k}) \rtimes_{v_M} \mathbb{R}$, with $(\mathbf{1},t)\cdot(f,0)\cdot(\mathbf{1},-t) = (f(\,\cdot\,+t),0)$ on G and $[(0,1),(\xi,0)] = (\mathcal{L}_{v_M}\xi, 0)$ on $\mathfrak{g}$. Because our principal bundles $\pi\colon P \to M$ are locally trivial and our actions T_M are locally free, every gauge group *locally* looks like the one derived from (M, T_M) as above. The extra data coming from (P,T) (over and above the information present in (M,T_M)) should be seen as *global* data, describing a *twist*.

A second important special case is $M = S^1$, where every principal fibre bundle is isomorphic to a bundle $P_\sigma \to S^1$, obtained by gluing the endpoints $\{0\}\times K$ and $\{2\pi\}\times K$ of $[0,2\pi]\times K$ together while twisting with a finite order automorphism $\sigma \in \mathrm{Aut}(K)$. The gauge algebra $\mathfrak{gau}_c(P_\sigma)$ is then isomorphic to the *twisted loop algebra*

$$\mathcal{L}_\sigma(\mathfrak{k}) := \{\xi \in C^\infty(\mathbb{R},\mathfrak{k})\,;\, \xi(t+2\pi) = \sigma(\xi(t))\}\,. \tag{1}$$

The adjective 'twisted' is only used if the class of σ in $\pi_0(\mathrm{Aut}(\mathfrak{k}))$, hence the principal K-bundle P_σ, is nontrivial.

3. Cocycles

For infinite dimensional Lie groups G modelled on a barrelled locally convex Lie algebra $\mathfrak{g}$, there is a bijective correspondence between projective unitary representations of G and a certain class (called 'regular' in [1]) of *linear* unitary representations (π, V) of a *central extension* $\mathbb{R} \to \mathfrak{g}^\sharp \to \mathfrak{g}$, where $\pi(1) = 2\pi i$ on $1 \in \mathbb{R} \hookrightarrow \mathfrak{g}^\sharp$. (A Lie algebra representation is called *unitary* if V has a nondegenerate Hermitean sesquilinear form, and all operators $\pi(\xi)$, $\xi \in \mathfrak{g}^\sharp$ are skew-symmetric.) Since *infinitesimal, linear* representations are much easier to handle than *global, projective* representations, our strategy is:

1) Classify the central extensions $\mathfrak{g}^\sharp \to \mathfrak{g}$.
2) Determine which ones come from projective unitary positive energy representations.
3) For a given extension $\mathfrak{g}^\sharp$, classify the unitary Lie algebra representations (π, V) with $\pi(1) = 2\pi i$ that integrate to a group representation.

Since equivalence classes of central extensions correspond to classes $[\omega] \in H^2(\mathfrak{g},\mathbb{R})$ in continuous Lie algebra cohomology, the first step comes down to calculating $H^2(\mathfrak{g},\mathbb{R})$. This was done in joint work with Christoph Wockel [3], where we showed that if we fix a volume form vol on M (which we take orientable), an equivariant connection ∇ on P, and a positive definite invariant bilinear form κ on $\mathfrak{k}$ (a multiple of the Killing form), then a class $[\omega_X]$ is uniquely determined by a distributon valued vector field $X \in \mathrm{Vec}(M) \otimes_{C^\infty(M)} \mathcal{D}'(M)$ with $\mathrm{div}(X) = 0$ and $L_{v_M}X + \mathrm{div}(v_M)X = 0$, by the formula

$$\omega_X(\xi,\eta) = \int_M \kappa(\xi, \nabla_X \eta)\mathrm{vol}\,.$$

Step two is handled in [2], where we show (the proof is inspired by a trick in [4]) that if ω_X comes from a projective positive energy representation, then $X = v_M \otimes \mu$, where μ is a T-invariant *measure* on M, and the cocycle is trivial on $\{0\} \rtimes_{v_M} \mathbb{R} \subseteq \mathfrak{g}$.

4. Step three: Localisation

In the context of loop algebras, cf. eqn. (1), where $v_M = \frac{d}{dt}$ lifts to v by the canonical flat connection with holonomy σ, we have $\mu = cdt$ with $c \geq 0$, so $X = c\frac{d}{dt}$ and $\mathfrak{g}^\sharp = (\mathbb{R} \oplus_\omega \mathcal{L}_\sigma(\mathfrak{k})) \rtimes_{\frac{d}{dt}} \mathbb{R}$ is the *affine Kac-Moody algebra* with cocycle $\omega(\xi, \eta) = c\int_0^{2\pi} \kappa(\xi, \partial_t \eta)dt$. If we normalise κ correctly, then the cocycle ω is integrable if and only if $c \in \mathbb{Z}$, the irreducible positive energy representations of $\mathfrak{g}^\sharp$ are precisely the *highest weight representations* $(\pi_\lambda, \mathcal{H}_\lambda)$ at dominant integral weight λ of $\mathfrak{g}^\sharp$ (see [5]), and the requirement $\pi_\lambda(1) = 2\pi i$ translates to the requirement that $c \in \mathbb{Z}$ be the *level* of λ.

Every point in M has a locally T-invariant open neighbourhood $U \simeq U_0 \times I$ (the interval $I \subseteq \mathbb{R}$ parametrizes the time direction and U_0 is a transversal slice) such that P is trivialisable on an open neighbourhood of $\overline{U}$. For such a neighbourhood, we can build the Banach Lie algebra $\overline{\mathfrak{gau}}_U := L^\infty(U_0, H^1(I, \mathfrak{k}))$, where the Hilbert–Lie algebra $H^1(I, \mathfrak{k})$ is the closure of $C_c^\infty(I, \mathfrak{k})$ for the inner product $(\xi, \eta)_{H^1} := \int_I \kappa(\xi, \eta)dt + \int_I (1 + t^2)\kappa(\partial_t \xi, \partial_t \eta)dt$. Since P trivialises over U, we have a continuous injection $\mathfrak{gau}_c(P|_U) \hookrightarrow \overline{\mathfrak{gau}}_U$, and the lynchpin of [2] is the *localisation lemma*, which says that the positive energy representation $(\mathtt{d}\rho, \mathcal{H}^\infty)$ extends to $\overline{\mathfrak{g}}_U$ as a continuous unitary Lie algebra representation on the space $\mathcal{H}_H^\infty$ of smooth vectors for the Hamilton operator. This allows one to apply $\mathtt{d}\rho$ on functions with a sharp cutoff in the direction transversal to v_M, which allows one to essentially cut the problem into little pieces.

5. Results

Using this localisation lemma, we proved in [2] that if M is compact and T_M (but not necessarily T!) is periodic, and proper and free as an $\mathbb{R}/\Lambda\mathbb{Z}$-action, then every irreducible smooth projective positive energy representation is an irreducible *evaluation representation on finitely many orbits.*

Evaluation representations are constructed as follows. If $\mathcal{O} \subseteq M$ is an orbit under time translation, then $\mathcal{O} \simeq S^1$ because T_M is periodic, so by the second example of Section 2, the restriction homomorphism $\mathrm{ev}_{\mathcal{O}} \colon \mathfrak{gau}(P) \to \mathfrak{gau}(P|_{\mathcal{O}})$ maps into a twisted loop algebra. If one picks finitely many distinct orbits $\mathcal{O}_i$, and for each orbit an irreducible unitary highest weight representation $(\pi_{\lambda_i}, \mathcal{H}_{\lambda_i})$ of the affine Kac-Moody algebra $(\mathbb{R} \oplus_\omega \mathfrak{gau}(P|_{\mathcal{O}_i})) \rtimes_{v_{\mathcal{O}_i}} \mathbb{R}$ with $\pi_{\lambda_i}(1, 0, 0) = 2\pi i$ (the level c is implicit in ω), then the *evaluation representation* associated to $\mathcal{O}_i$ and $(\pi_{\lambda_i}, \mathcal{H}_{\lambda_i})$ is the Hilbert space $\mathcal{H} := \bigotimes_{i=1}^n \mathcal{H}_{\lambda_i}$ with $\mathfrak{g}$-action given by $\pi(\xi) = \sum_{i=1}^n \mathbf{1} \otimes \ldots \otimes \pi_{\lambda_i}(\xi|_{\mathcal{O}_i}) \otimes \ldots \mathbf{1}$, and where, similarly, the Hamiltonian $H := -i\pi((0, 0, 1))$ is the sum of the Hamiltonians of the individual Kac–Moody algebras.

We stress that the compactness and periodicity assumptions only enter at a relatively late stage of the proof, and we expect our methods to apply in more general situations. For example, consider the 1-point compactification S^4 of $\mathbb{R}^4$ with principal K-bundle $P \to S^4$. The action of the Poincaré group $\mathrm{SO}(3,1) \ltimes \mathbb{R}^4$ yields a time translation which is proper and free on $\mathbb{R}^4$, but has fixed point ∞ on S^4. I expect (the proof is not yet sufficiently rigorous to use the term 'theorem') that every projective unitary representation of $\mathrm{Gau}(P)_0$ that extends to a positive energy representation of $\mathfrak{gau}_c(P) \rtimes (\mathfrak{so}(3,1) \ltimes \mathbb{R}^4)$ factors through a $\widetilde{K}$-representation by evaluating in ∞.

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Reflection positivity as it arises for operators in Hilbert space, in representation theory, in stochastic analysis, and in physics

Palle E.T. Jorgensen

While the study of reflection positivity began in mathematical physics in the 1970s (the Osterwalder-Schrader theory, and Euclidean fields), it has acquired a life of its own in independent questions for operators in Hilbert space, the study of spectral theory, in representation theory (analytic continuation of unitary representations of Lie groups), in stochastic analysis, and in physics. The details of this were reviewed in the presentation by myself as well as the other speakers at the workshop. In the talk Jorgensen referred to both his joint work with G. Olafsson (see [2], [3]), and more recently with K.H. Neeb; cited by the other speakers. As well as to the math physics literature, see e.g., [7], [8], [9], [4], [5], [6].

The tools from: (i) operators in Hilbert space, from (ii) the theory of unitary representations and their c-duals; and from (iii) stochastic processes will be introduced in steps (i)→(ii)→(iii).

In (i), we have the following setting: $\mathcal{E}$: a fixed Hilbert space; $e_\pm, e_0$: three given projections in $\mathcal{E}$; R: a fixed period-2 unitary: $\mathcal{E} \to \mathcal{E}$.

We assume that:

$$(1) \qquad \text{(Refl)} \quad Re_+ = e_-R, \text{and } Re_0 = e_0$$

Definition 1.

(M) Markov: $e_+e_0e_- = e_+e_-$ *and (RF) Osterwalder-Schrader:* $e_+Re_+ \geq 0$.

Lemma 2. *(M) ⇒ (RF).*

Proof. Assume (M), then

$$e_+ R\, e_+ \underset{\text{by } (Refl)}{=} e_+\, e_-\, R \underset{\text{by } (M)}{=} e_+\, e_0\, e_- R = e_+\, e_0\, R\, e_+ = e_+\, e_0\, e_+ \geq 0. \qquad \square$$

Among the new results presented at the workshop are the following:

Lemma 3. *Let e_+ and R be as above, and suppose $e_+ R e_+ \geq 0$, and set*

$$\mathcal{N} := \{\psi \in \mathcal{E} | \langle \psi, R\psi \rangle = 0\}, \text{ and}$$

$$q : \mathcal{E}_+ \longrightarrow \mathcal{E}_+/\mathcal{N} \longrightarrow \widehat{\mathcal{E}} := compl.\left(\tfrac{\mathcal{E}_+}{\mathcal{N}}\right)$$
$$\psi \mapsto \psi + \mathcal{N}$$

(i) *Then q is contractive from $\mathcal{E}_+$ to $\widehat{\mathcal{E}}$, and*
(ii) $q^* q = e_+ R e_+$.

If further $e_0 \leq e_\pm$, then $q^* q = e_0$.

(iii) *There is a quantization mapping: $b \longrightarrow \widehat{b}$ from the algebra $\mathcal{B}(\mathcal{E}_+, \mathcal{N})$ of bounded operators on $\mathcal{E}_+$ preserving $\mathcal{N}$ into $\mathcal{B}(\widehat{\mathcal{E}})$ such that:*

$$qb = \widehat{b}\, q,\ e_0 b = q^* \widehat{b}\, q,\ \widehat{(b_1 b_2)} = \widehat{b_1} \widehat{b_2},\ \forall b_1, b_2 \in \mathcal{B}(\mathcal{E}_+, \mathcal{N}).$$

(iv) *If $RbR = b^*$, then $(\widehat{b})^* = \widehat{b}$.*

Theorem 4 (Jorgensen-Neeb-Olafsson)**.** *Let G be a locally compact group with a closed semigroup $S \subset G$, $e \in S$, and $S^{-1} \cup S = G$. Let $\mathcal{H}$ be a Hilbert space, and P a representation of S acting in $\mathcal{H}$. Assume $v_0 \in \mathcal{H}$, $\|v_0\| = 1$, satisfies $P(s)v_0 = v_0$ for all $s \in S$. Let Q be a compact Hausdorff space, and let $\mathcal{A} \simeq C(Q)$ be an abelian C^*-algebra acting on $\mathcal{H}$. Assume that for all n, all $x_1, x_2, \cdots, x_n \in G$ such that*

$$x_1 \in S, x_1^{-1} x_2 \in S, \cdots, x_{n-1}^{-1} x_n \in S, \tag{2}$$

and all $f_i \in \mathcal{A}_+$, $1 \leq i \leq n$, we have

$$\langle v_0, f_1 P(x_1^{-1} x_2) f_2 \cdots P(x_{n-1}^{-1} x_n) f_n v_0 \rangle \geq 0 \tag{3}$$

Set $\Omega := Q^G$.

Let μ_F, $F = \{x_1, x_2, \cdots, x_n\}$, be the measure on $Q^n = \underbrace{Q \times \cdots \times Q}_{n \text{ times}}$ specified by

$$\int_{Q^n} f_1 \otimes \cdots \otimes f_n d\mu_F = \langle v_0, f_1 P(x_1^{-1} x_2) f_2 \cdots P(x_{n-1}^{-1} x_n) f_n v_0 \rangle \tag{4}$$

Let $\mathcal{F}_S = \{F |$ specified by conditions $(2)\}$.

For measures μ on $\Omega := Q^G$ ($=$ all functions $w : G \to Q$), the pathspace, we denote by π_F^*, $F \in \mathcal{F}_S$ the pull-back $\mu_F^*(\mu)$ via the coordinate projection $\pi_F : \Omega \to Q^F$.

Then there is a unique measure μ on Ω such that

$$\pi_F^*(\mu) = \mu_F \text{ for all } F \in \mathcal{F}_S. \tag{5}$$

Moreover, the solution μ to (5) is G-invariant.

The sigma-algebra for μ is the sigma-algebra of subsets of Ω generated by the cylinder sets.

Proof. We refer to our papers for a detailed exposition of the proof details. □

Glossary/terms used:

Modular theory (MT) used here refers to the theory of von Neumann algebras, and is also called Tomita-Takesaki theory. In von Neumann algebra theory, MT was (since the 1970s) the key building block in our understanding of type III factors. But MT also lies at the foundation of rigorous formulations of equilibrium states (temperature-states) in statistical mechanics. Finally, Hans Borchers' formulation of local quantum fields makes critically use of the one-parameter modular group of automorphisms from MT.

Super: The notion of superalgebra refers to a Z_2-grading. It has been used extensively in the study of (super) Lie algebras arising in physics. Lie superalgebras: The even elements of the superalgebra correspond to bosons, and odd elements to fermions.

Free fields: Quantum field theory (QFT) is a theoretical framework for constructing quantum mechanical models of subatomic particles and fields. A QFT treats particles as excited states of an underlying physical field. The case of free fields encompasses the mathematical model before interaction is added.

Borchers-triples (BT): The term "Borchers-triple" was suggested by Buchholz, it refers to a triple (M, U, v) where M is a von Neumann algebra in a Hilbert space $\mathcal{H}, U$ is a unitary representation of space-time, satisfying the spectrum-condition and acting by conjugation on M, v is unit vector in $\mathcal{H}$ (vacuum) fixed by U, and cyclic and separating for M. The spectrum-condition refers to a prescribed covariance system for (M, U). The triples are used in our understanding of local quantum fields, and they are studied with the use of modular theory.

Borchers-local fields: von Neumann algebras indexed by local regions in space time.

Our use of "*Lax-Phillips model*" refers to Lax-Phillips scattering theory, but applies more generally to any unitary one-parameter groups in Hilbert space which satisfy certain axioms for an outgoing subspace; and it allows us to realize such one-parameter groups, up to unitary equivalence, as groups of translation operators.

Wightman QFT-axioms: An attempt by Arthur Wightman in the 1950s, at a mathematically rigorous formulation of quantum field theory. While the Wightman fields are operator valued distributions, by contrast Euclidean fields are algebras of certain random variables; hence commutative algebras. The OS-axioms form a link between them, via a subtle analytic continuation and renormalization.

Commentary: In the axioms for PR, the reflection R typically takes the form of a unitary period-2 operator in Hilbert space. However, in order to get a more natural fit between RP and modular theory, it is more natural that R be anti-unitary. The operator J from modular theory is anti-unitary. If M is a von-Neumann algebra with a chosen faithful, normal state s , then there is a corresponding J such that $JMJ = M'$ (the commutant of M;) and, fixing β there is a one-parameter group of automorphisms of M which has the state s as a β-KMS state.

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Reflection Positivity and Operator Algebras in Quantum Field Theory

Gandalf Lechner

In this talk, some operator-algebraic aspects of quantum field theory (QFT) were reviewed, and their links/similarities to reflection positivity was discussed.

The basic setting is that of modular theory, either of a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ with a cyclic separating vector $\Omega \in \mathcal{H}$ (algebraic version), or of a closed real standard subspace $H \subset \mathcal{H}$ of a Hilbert space (spatial version). We first recalled the basic definitions in this context – in particular the Tomita operator S associated with $(\mathcal{M}, \Omega)$ and $H \subset \mathcal{H}$, respectively, its polar decomposition, and the Tomita-Takesaki Theorem as well as the KMS condition. It was also recalled how to proceed from the algebraic to the spatial version (by $H := \overline{\mathcal{M}_{\mathrm{sa}}\Omega}$) and from the spatial version to the algebraic version (by second quantization).

In QFT, one is usually interested in situations with additional structure, such as group actions and/or specific subalgebras/-spaces. In the simplest case – pertaining to quantum fields localized on a half line – this amounts to considering a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ (respectively, a standard subspace $H \subset \mathcal{H}$) and a unitary strongly continuous $\mathbb{R}$-action T on $\mathcal{H}$ such that $T(t)\mathcal{M}T(-t) \subset \mathcal{M}$ for $t \geq 0$ and $T(t)\Omega = \Omega$ for all $t \in \mathbb{R}$ (respectively, $T(t)H \subset H$ for $t \geq 0$). A theorem by Borchers [1] then states that in case the generator P of this representation is positive/negative, $\pm P > 0$, then the modular data J, Δ satisfy

$$\Delta^{it}T(s)\Delta^{-it} = T(e^{\mp 2\pi t} \cdot s), \qquad JT(s)J = T(-s)\,.$$

In other words, T extends to a (anti-) unitary representation of the "$ax + b$ group". It is also worth mentioning that this situation is very similar to the Lax-Phillips situation of an "outgoing subspace", the difference being that the space under consideration is only real linear here. In the non-degenerate spatial case, i.e. if there are no T-invariant vectors in H, and the $ax + b$ representation is irreducible, there exists a unique standard pair (H, T), which can for example be described on $\mathcal{H} = L^2(\mathbb{R}_+, dp/p)$, with

$$(T(t)\psi)(p) = e^{ipt} \cdot \psi(p), \qquad (\Delta^{it}\psi)(p) = \psi(e^{-2\pi t}p), \qquad (J\psi)(p) = \overline{\psi(p)}.$$

Here H consists of all functions in $\mathcal{H}$ which are boundary values of Hardy-type functions on the upper half plane, satisfying a reality condition [2].

If one considers the two-dimensional situation, i.e. asks for an $\mathbb{R}^2$-action $(x_+, x_-) \mapsto T(x_+, x_-)$, with both generators positive, and which acts by endomorphisms on $\mathcal{M}$ (respectively, H) for $x_+ \geq 0$, $x_- \leq 0$ (physically, the coordinates $x_\pm \in \mathbb{R}$ parametrize the two light rays through the origin of two-dimensional Minkowski space), one obtains by Borchers' theorem an extension of T from $\mathbb{R}^2$ to the two-dimensional proper Poincaré group.

On the QFT side, Borchers triples (consisting of a von Neumann algebra with cyclic separating vector and a half-sided $\mathbb{R}^2$-action T as above), respectively standard pairs (consisting of a real standard subspace with half-sided $\mathbb{R}^2$-action) can be considered as "germs" of local, covariant models of QFT. In fact, by using the group action and intersections, one can proceed from the single algebra $\mathcal{M}$ (respectively, real subspace H) to a net of algebras/real subspaces, indexed by the open subsets of $\mathbb{R}^2$, and enjoying natural locality and covariance properties. This motivates the analysis of Borchers triples, and since Euclidean formulations have proven to be a valuable tool in constructive QFT in the past, the question emerges whether useful Euclidean realizations of Borchers triples and / or standard pairs exist.

The connection of representations of the Poincaré group to representations of the Euclidean group is known to be closely linked to reflection positivity (see, for example [3, 4]). What is required in addition here is a Euclidean datum representing the half-sided algebra/subspace. The talk ended with a sketch of how the approach of Schlingemann [5] might be useful in this context, and pointed out the double role of reflections (time reflection as usual, and space-time reflection, represented by the modular conjugation J) present in this setting.

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Examples of reflection positive representations of semisimple groups

Jan Möllers

The first use of reflection positivity in the context of unitary representation theory of semisimple groups is due to Robert Schrader. In his paper [4] he relates complementary series representations of $\mathrm{SL}(2n,\mathbb{C})$ to certain unitary representations of the product group $\mathrm{SU}(n,n)\times\mathrm{SU}(n,n)$ (or rather its universal cover). This concept was later picked up by Jorgensen–Ólafsson [2] and Neeb–Ólafsson [3] and applied to different pairs of semisimple groups. In this talk we provide a unified framework into which all these examples fit.

This uniform treatment can be applied to obtain a correspondence between certain unitary representations of pairs of groups which were studied by Enright [1] even before Schrader's paper [4]. We remark that Enright does not use or even mention reflection positivity at all.

1. The possible pairs $(\mathfrak{g},\mathfrak{g}^c)$

Let $\mathfrak{g}$ be a simple real Lie algebra and τ an involution of $\mathfrak{g}$. Write $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ for the decomposition of $\mathfrak{g}$ into τ-eigenspaces to eigenvalues $+1$ and -1. We assume that the c-dual Lie algebra

$$\mathfrak{g}^c := \mathfrak{h}+i\mathfrak{q}\subseteq\mathfrak{g}_{\mathbb{C}}$$

is Hermitian of tube type, i.e. the corresponding Riemannian symmetric space G^c/K^c is a bounded symmetric domain of tube type. We discuss examples where complementary series representations of $\mathfrak{g}$ can be related to unitary highest weight representations of $\mathfrak{g}^c$.

2. The pair $(\mathfrak{so}(n+1,1),\mathfrak{so}(2,n))$

This example is for $n=1$ due to Jorgensen–Ólafsson [2, Introduction] and for $n>1$ due to Neeb–Ólafsson [3, Section 6].

Let $G=\mathrm{O}(n+1,1)$ and $\mathfrak{g}=\mathfrak{so}(n+1,1)$ its Lie algebra. The (spherical) complementary series representations π_s ($s\in(-\frac{n}{2},\frac{n}{2})$) of the group G can be realized on Hilbert spaces $\mathcal{E}_s$ of distributions on $\mathbb{R}^n$ endowed with the inner product

$$\langle u,v\rangle_s=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x-y|^{-2s-n}u(x)\overline{v(y)}\,dx\,dy.$$

(This expression makes sense at least for $s\in(-\frac{n}{2},0)$.) The unitary action π_s of G on $\mathcal{E}_s$ is induced by the conformal action of G on $\mathbb{R}^n$.

Consider the unitary involution θ on $\mathcal{E}_s$ given by

$$\theta u(x)=|x|^{2s-n}u\left(\frac{x}{|x|^2}\right),\qquad x\in\mathbb{R}^n.$$

The involution θ is in fact given by $\theta = \pi_s(w)$ for a certain element $w \in G$ of order 2. This element defines an (inner) involution τ of G by $\tau(g) = wgw$ and the corresponding c-dual $\mathfrak{g}^c$ of the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(n,2)$.

Let $\mathcal{E}_{s,+} \subseteq \mathcal{E}_s$ be the subspace

$$\mathcal{E}_{s,+} = \{u \in \mathcal{E}_s : \operatorname{supp} u \subseteq \mathcal{D}\},$$

where $\mathcal{D} = \{x \in \mathbb{R}^n : |x| < 1\}$ is the unit disc. For $u \in \mathcal{E}_{s,+}$ a short computation shows that

$$\langle u, v\rangle_{s,\theta} := \langle \theta u, v\rangle_s = \int_{\mathcal{D}}\int_{\mathcal{D}} (1 - 2x\cdot y + |x|^2|y|^2)^{-s-\frac{n}{2}} u(x)\overline{v(y)}\,dx\,dy.$$

We note that the unit disc $\mathcal{D}$ is the intersection of the *Lie ball*

$$\mathcal{D}_{\mathbb{C}} = \{z = x + iy \in \mathbb{C}^n : |x|^2 + |y|^2 + 2\sqrt{|x|^2|y|^2 - (x\cdot y)^2} < 1\}$$

with $\mathbb{R}^n \subseteq \mathbb{C}^n$, and that the kernel function $(1-2x\cdot y+|x|^2|y|^2)^{-s-\frac{n}{2}}$ is identified with the restriction of a power of the Bergman kernel $(1 - 2z\cdot\overline{w} + z^2\overline{w}^2)^{-n}$ of the bounded symmetric domain $\mathcal{D}_{\mathbb{C}}$ to the totally real subdomain $\mathcal{D}$. The bounded symmetric domain $\mathcal{D}_{\mathbb{C}}$ is the Riemannian symmetric space G^c/K^c and it is well known that the corresponding power of the Bergman kernel is positive definite on $\mathcal{D}_{\mathbb{C}}$ (and hence on the totally real submanifold $\mathcal{D}$) if and only if s is contained in the so-called *Berezin–Wallach set* $\{-\frac{n}{2}\} \cup [-1, \infty)$.

Forming the intersection of the parameter ranges $(-\frac{n}{2}, 0)$ and $\{-\frac{n}{2}\} \cup [-1, \infty)$ this yields the following theorem:

Theorem (Jorgensen–Ólafsson [2], Neeb–Ólafsson [3]). (1) *Let $s \in (-\frac{n}{2}, 0)$, then the form $\langle\cdot,\cdot\rangle_{s,\theta}$ is positive semidefinite on $\mathcal{E}_{s,+}$ if and only if $s \in [-1, 0)$. In this case the completion of $\mathcal{E}_{s,+}/\mathcal{N}$ where $\mathcal{N}$ is the null space of $\langle\cdot,\cdot\rangle_{s,\theta}$ will be denoted by $\widehat{\mathcal{E}}_s$.*

(2) *For $s \in [-1, 0)$ we obtain a unitary representation $\widehat{\pi}_s$ of the universal cover G^c of $\mathrm{O}(n,2)$ on $\widehat{\mathcal{E}}_s$. The representations of G^c obtained in this way are unitary highest weight representations (or analytic continuations of holomorphic discrete series).*

We remark that instead of the reflection θ at the unit sphere one can as well consider the time reflection

$$\tilde{\theta}u(x_1, \ldots, x_{n-1}, x_n) = u(x_1, \ldots, x_{n-1}, -x_n)$$

on the subspace

$$\tilde{\mathcal{E}}_{s,+} = \{u \in \mathcal{E}_s : \operatorname{supp} u \subseteq \mathbb{R}^n_+\},$$

where $\mathbb{R}^n_+ = \{x_n > 0\}$. Since θ and $\tilde{\theta}$ are unitarily equivalent via the action π_s of G this yields unitarily equivalent Hilbert spaces and representations.

3. The pair $(\mathfrak{gl}(2n,\mathbb{C}), \mathfrak{u}(n,n) \times \mathfrak{u}(n,n))$

The following example is (in a slightly different way) due to Schrader [4].

We consider the group $G = \mathrm{GL}(2n,\mathbb{C})$. Corresponding to the parabolic subgroup for the partition $2n = n + n$ there are complementary series representations

π_s ($s \in (-1,1)$) of G on a Hilbert space $\mathcal{E}_s$ of distributions on $M(n\times n,\mathbb{C})$ endowed with the inner product

$$\langle u,v\rangle_s = \int_{M(n\times n,\mathbb{C})}\int_{M(n\times n,\mathbb{C})} |\det_{\mathbb{C}}(x-y)|^{-2s-2n} u(x)\overline{v(y)}\,dx\,dy.$$

(This expression makes sense at least for $s \in (-1,0)$.) In this case the involution θ is given by

$$\theta u(x) = |\det_{\mathbb{C}} x|^{2s-2n} f((x^*)^{-1}), \qquad x \in M(n\times n,\mathbb{C})$$

and the corresponding subspace $\mathcal{E}_{s,+} \subseteq \mathcal{E}_s$ is given by

$$\mathcal{E}_{s,+} = \{u \in \mathcal{E}_s : \operatorname{supp} u \subseteq \mathcal{D}\},$$

where $\mathcal{D} = \{x \in M(n\times n,\mathbb{C}) : \|x\| < 1\}$, the operator norm $\|\cdot\|$ being induced from the Euclidean norm on $\mathbb{C}^n$. A short computation for $u \in \mathcal{E}_{s,+}$ shows

$$\langle u,u\rangle_{s,\theta} = \langle \theta u, u\rangle_s = \int_{\mathcal{D}}\int_{\mathcal{D}} |\det_{\mathbb{C}}(1-x^*y)|^{-2s-2n} u(x)\overline{u(y)}\,dx\,dy.$$

Now note that $\mathcal{D}$ embeds into $\mathcal{D}\times\overline{\mathcal{D}}$ as a totally real subspace by $x \mapsto (x,\overline{x})$ and via this embedding the kernel

$$|\det_{\mathbb{C}}(1-x^*y)|^{-2s-2n} = \det_{\mathbb{C}}(1-x^*y)^{-s-n}\cdot \det_{\mathbb{C}}(1-x^T\overline{y})^{-s-n}$$

is the restriction of the product of powers of the Berman kernels of $\mathcal{D}$ and $\overline{\mathcal{D}}$. The powers $\det_{\mathbb{C}}(1-x^*y)^{-s-n}$ are positive definite on $\mathcal{D}$ if and only if s is contained in the Berezin–Wallach set $\{-n,\dots,-1\}\cup(-1,\infty)$ and hence reflection positivity applies for $s\in(-1,0)$ and yields unitary representations $\widehat{\pi}_s$ of the universal cover of $G^c = \mathrm{U}(n,n)\times\mathrm{U}(n,n)$ which are tensor products of unitary highest weight and lowest weight representations.

4. The pairs $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{h}\times\mathfrak{h})$

Generalizing the pair $(\mathfrak{gl}(2n,\mathbb{C}), \mathfrak{u}(n,n)\times\mathfrak{u}(n,n))$ one can consider $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{h}\times\mathfrak{h})$. Here $\mathfrak{h}$ is any Hermitian Lie algebra of tube type, $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ its complexification viewed as a real Lie algebra. Then $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{h}_{\mathbb{C}}\times\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{g}^c = \mathfrak{h}\times\mathfrak{h}$ is another real form of $\mathfrak{g}_{\mathbb{C}}$.

These cases are discussed by Enright [1] without using reflection positivity. However, employing the same machinery as in Section 3 one can obtain the same results as Enright does in the spirit of reflection positivity.

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From the path integral to the Hamiltonian formalism in a diffeomorphism–invariant context

José Mourão

(joint work with Abhay Ashtekar, Jerzy Lewandowski, Donald Marolf, Thomas Thiemann)

The Osterwalder–Schrader axioms allow us to construct the Hamiltonian Lorentzian formalism from the rigorous Euclideanized path integral in a quantum scalar field theory on $\mathbb{R}^{d+1}$ [7, 11, 12]. We report on the results obtained in [4, 5] on an extension of the Osterwalder–Schrader axioms to theories invariant under $\mathrm{Diff}(M, s)$, the group of diffeomorphisms of space–time M, assumed to be diffeomorphic to $\mathbb{R} \times \Sigma$, preserving the structure s.

The ultimate goal would of course be to develop a formalism applicable to Quantum Gravity but a very nice instructive example is given by Yang–Mills theory on two dimensions, YM_2. The classical Yang-Mills action

$$S_{YM}(A) = -\frac{1}{g^2} \int_M trF \wedge \star F,$$

is invariant under $\mathrm{Diff}(M, Ar)$, the infinite–dimensional group of area preserving difeomorphisms.

In [4] we used the realization of the Ashtekar–Isham compactification, $\overline{\mathcal{A}/\mathcal{G}}$ [1], of the infinite–dimensional space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations on M, as the projective limit of spaces of connections modulo gauge transformations on finite graphs (homeomorphic to G^n/Ad_G) [2, 3, 10, 15, 16] and the calculation of the expectation values of multiple loop Wilson variables (products of traces of holonomies)

$$< T_{\alpha_1} \ldots T_{\alpha_n} >= \int_{\overline{\mathcal{A}/\mathcal{G}}} T_{\alpha_1}(A) \ldots T_{\alpha_n}(A)\, d\mu_{YM_2} \ , \tag{1}$$

as the continuum limit of a lattice regularized model, to define a unique σ–additive probability measure, μ_{YM_2}, on $\overline{\mathcal{A}/\mathcal{G}}$. The expectation values (1) were also calculated in the framework of rigorous measures on the affine space of random connections on $\mathbb{R}^2$, with linear gauge fixing, in [6, 8] and extended to arbitrary Riemann surfaces in [13, 14]. For measures on $\overline{\mathcal{A}/\mathcal{G}}$ the results of [4] were extended to arbitrary Riemann surfaces in [9].

By identifying time with the first coordinate in $\mathbb{R}^2$ and in $\mathbb{R} \times S^1$, we showed in [4] that the measure μ_{YM_2} defined by (1) satisfies a natural generalization, to quantum theories of connections, of the Osterwalder–Schrader axioms including reflection positivity. The Osterwalder–Schrader reconstruction theorem is then valid and, in accordance with the classical Hamiltonian formulation, the physical Hilbert space is trivial if $M = \mathbb{R}^2$ and is naturally isomorphic to the space of Ad–invariant, square integrable functions on G, $L^2(G, dx)^{Ad}$, where dx is the Haar measure, if $M = \mathbb{R} \times S^1$. In the latter case the Hamiltonian operator obtained from the reconstruction theorem is

$$\widehat{H} = -\frac{g^2}{2} L\,\Delta,$$

where L is the length of S^1 and Δ is the Laplacian on G, corresponding to the bi-invariant metric.

The invariance of μ_{YM_2} under the infinite dimensional group $\mathrm{Diff}(M, Ar)$ motivated us to extend the Osterwalder–Schrader axioms to the context of general quantum field theories invariant under $\mathrm{Diff}(M, s)$, for a space–time M diffeomorphic to $\mathbb{R} \times \Sigma$, with a fixed background structure s [5]. In quantum gravity one expects the path integral measure to have $\mathrm{Diff}(M)$ acting as gauge if Σ is compact without boundary but, for example, in asymptoticaly flat or anti-De Sitter space-times M, one expects the Euclideanized path integral measure to be invariant only under diffeomorphisms preserving the Euclideanized asymptotic structure and only those asymptotically trivial diffeomorphisms should act as gauge. In these cases the asymptotic structure will play the role of background structure s.

To formulate the generalized axioms, for a theory with background structure s on $M \cong \mathbb{R} \times \Sigma$, we say that a foliation F of space-time is compatible with s if it belongs to the following subset of the set $B = \mathrm{Diff}(\mathbb{R} \times \Sigma, M)$ of diffeomorphisms from $\mathbb{R} \times \Sigma$ to M,

$$\mathcal{F} = \left\{F \in B : \theta^F = F \circ \theta \circ F^{-1}, T_t^F = F \circ T_t \circ F^{-1} \in \mathrm{Diff}(M, s), \forall t \in \mathbb{R}\right\}, \tag{2}$$

where $T_t(s, x) = (s+t, x)$ and $\theta(t, x) = (-t, x)$. Reflection positivity of the relevant correlation functions was then required for every s–compatible foliation F and the reconstruction of the Hamiltonian formalism proved for every such F. Different choices of compatible foliations may lead to inequivalent Hamiltonian theories as was shown to be the case for Yang-Mills theory on $\mathbb{R} \times S^1$. Two equivalence relations, weak and strong, were introduced on $\mathcal{F}$. Strongly equivalent foliations lead to equivalent Hamiltonian theories, while weakly equivalent foliations have naturally isomorphic physical Hilbert spaces but possibly inequivalent Hamiltonian theories.

Besides YM_2 on $\mathbb{R}^2$ and on $\mathbb{R} \times S^1$ we showed that the extension of the Osterwalder–Schrader axioms is also valid for $2+1$ gravity and BF theories.

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A representation theoretic perspective on reflection positivity

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(joint work with Gestur Ólafsson)

We report on a long-term project with G. Ólafsson ([NO14a, NO14b, NO15, MNO14]). It aims at a better understanding of reflection positivity, a basic concept in constructive quantum field theory ([GJ81, JOl00, JR07a, JR07b]). Originally, reflection positivity, also called Osterwalder–Schrader positivity (or OS positivity), was conceived as a requirement on the euclidean side to establish a duality between euclidean and relativistic quantum field theories ([OS73]). It is closely related to "Wick rotations" or "analytic continuation" in the time variable from the real to the imaginary axis.

The underlying fundamental concept is that of a *reflection positive Hilbert space*, introduced in [NO14a]. This is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, where $\mathcal{E}$ is a Hilbert space, $\theta : \mathcal{E} \to \mathcal{E}$ is a unitary involution and $\mathcal{E}_+$ is a closed subspace of $\mathcal{E}$ which is θ-positive in the sense that the hermitian form $\langle u, v\rangle_\theta := \langle \theta u, v\rangle$ is positive semidefinite on $\mathcal{E}_+$. Let $\mathcal{N} := \{v \in \mathcal{E}_+ : \langle \theta v, v\rangle = 0\}$, write $\widehat{\mathcal{E}}$ for the Hilbert space completion of the quotient $\mathcal{E}_+/\mathcal{N}$ with respect to $\|v\|_\theta := \sqrt{\langle \theta v, v\rangle}$ and $q\colon \mathcal{E}_+ \to \widehat{\mathcal{E}}, v \mapsto \widehat{v}$ for the canonical map. If

$$T\colon \mathcal{D}(T) \subseteq \mathcal{E}_+ \to \mathcal{E}_+$$

is a linear or antilinear operator with $T(\mathcal{N} \cap \mathcal{D}(T)) \subseteq \mathcal{N}$, then there exists a well-defined operator

$$\widehat{T}\colon \mathcal{D}(\widehat{T}) \subseteq \widehat{\mathcal{E}} \to \widehat{\mathcal{E}} \quad \text{defined by} \quad \widehat{T}(\widehat{v}) = \widehat{Tv}, \quad v \in \mathcal{D}(T).$$

To relate this to group representations, let us call a triple (G, H, τ) a *symmetric Lie group* if G is a Lie group, τ is an involutive automorphism of G and H is an open subgroup of the group G^τ of τ-fixed points. Then the Lie algebra $\mathfrak{g}$ of G decomposes into τ-eigenspaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and we obtain the *Cartan dual Lie algebra* $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$. If (G, H, τ) is a symmetric Lie group and $(\mathcal{E}, \mathcal{E}_+, \theta)$ a reflection positive Hilbert space, then we say that a unitary representation $\pi\colon G \to \mathcal{E}$ is *reflection positive with respect to* (G, H, τ) if the following three conditions hold:

(RP1) $\pi(\tau(g)) = \theta\pi(g)\theta$ for every $g \in G$.
(RP2) $\pi(h)\mathcal{E}_+ = \mathcal{E}_+$ for every $h \in H$.
(RP3) There exists a subspace $\mathcal{D} \subseteq \mathcal{E}_+ \cap \mathcal{E}^\infty$ whose image $\widehat{\mathcal{D}}$ in $\widehat{\mathcal{E}}$ is dense such that $\mathtt{d}\pi(X)\mathcal{D} \subset \mathcal{D}$ for all $X \in \mathfrak{q}$.

A typical source of reflection positive representations are the representations $(\pi_\varphi, \mathcal{H}_\varphi)$ obtained via GNS construction [NO14a] from τ-invariant positive definite functions $\varphi\colon G \to B(\mathcal{V})$, respectively the kernel $K(x, y) = \varphi(xy^{-1})$, where $\mathcal{V}$ is a Hilbert space. If $G^+ \subseteq G$ is an open subset with $G^+H = G^+$, then φ is called *reflection positive for* (G, G^+, τ) if the kernel $Q(x, y) = \varphi(x\tau(y)^{-1})$ is positive definite on G^+. For $\mathcal{E} = \mathcal{H}_\varphi$, the subspace $\mathcal{E}_+$ is generated by the functions $K(\cdot, y)v$, $y \in G^+, v \in \mathcal{V}$, and $\widehat{\mathcal{E}}$ identifies naturally with the Hilbert space $\mathcal{H}_Q \subseteq \mathcal{V}^{G^+}$ (cf. [NO14a, Prop.1.11]). If the kernels $\langle Q(x, y)v, v\rangle$ are smooth for v in a dense subspace of $\mathcal{V}$, then (RP1-3) are readily verified.

If π is a reflection positive representation, then $\pi^c_H(h) := \widehat{\pi(h)}$ defines a unitary representation of H on $\widehat{\mathcal{E}}$. However, we would like to have a unitary representation π^c of the simply connected Lie group G^c with Lie algebra $\mathfrak{g}^c$ on $\widehat{\mathcal{E}}$ extending π^c_H in such a way that the derived representation is compatible with the operators $i\widehat{\mathtt{d}\pi(X)}$, $X \in \mathfrak{q}$, that we obtain from (RP3) on a dense subspace of $\widehat{\mathcal{E}}$. If such a representation exists, then we call $(\pi, \mathcal{E})$ a *euclidean realization* of the representation $(\pi^c, \widehat{\mathcal{E}})$ of G^c. Sufficient conditions for the existence of π^c have been developed in [MNO14]. The prototypical pair (G, G^c) consists of the euclidean motion group $\mathbb{R}^d \rtimes \mathrm{O}_d(\mathbb{R})$ and the simply connected covering of the Poincaré group $\mathcal{P}^\uparrow_+ = \mathbb{R}^d \rtimes \mathrm{SO}_{1,d}(\mathbb{R})_0$.

In [NO14b] we study reflection positive one-parameter groups and hermitian contractive semigroups as one key to reflection positivity for more general symmetric Lie groups and their representations. Here a crucial point is that, for every unitary one-parameter group $U^c_t = e^{itH}$ with $H \geq 0$ on the Hilbert space $\mathcal{V}$, we obtain by $\varphi(t) := e^{-|t|H}$ a $B(\mathcal{V})$-valued function on $\mathbb{R}$ which is reflection positive for $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_\mathbb{R})$ and which leads to a natural euclidean realization of U^c. From this we derive that all representations of the $ax+b$-group, resp., the Heisenberg group which satisfy the positive spectrum condition for the translation group, resp., the center, possess euclidean realizations.

In [NO15] we explore reflection positive functions $\varphi\colon \mathbb{T} \to B(\mathcal{V})$ for $(\mathbb{T}, \mathbb{T}^+, \tau)$, where $\tau(z) = z^{-1}$ and $\mathbb{T}^+$ is a half circle. This leads us naturally to anti-unitary involutions, an aspect that did not show up for triples (G, G^+, τ), where G^+ is a semigroup. In particular, we characterize those unitary one-parameter groups $(U^c, \mathcal{H})$ which admit euclidean realizations in this context as those for which there exists an anti-unitary involution J commuting with U^c. Any such pair (J, U^c) with $U^c_t = e^{itH}$ can be encoded in the pair (J, Δ), where $\Delta = e^{-\beta H}$ is positive selfadjoint with $J\Delta J = \Delta^{-1}$, a relation well-known from Tomita–Takesaki theory. Finally, we describe a link between KMS states for C^*-dynamical systems $(\mathcal{A}, \mathbb{R}, \alpha)$.

It would be very intersting to develop the representation theoretic side of reflection positivity under the presence of anti-unitary involutions which occur in many constructions in Quantum Field Theory (see [BS04, p. 627], [Bo92], [BGL02]). Here are some corresponding remarks. If $(\mathcal{E}, \mathcal{E}_+, \theta)$ is a reflection positive Hilbert space and $J\colon \mathcal{E} \to \mathcal{E}$ an antiunitary involution satisfying $J\mathcal{E}_+ = \mathcal{E}_+$ and $J\theta = \theta J$, then J introduces an antiunitary involution $\widehat{J}$ on $\widehat{\mathcal{E}}$. If, in addition, $(U_t)_{t\in\mathbb{R}}$ is a reflection positive one-parameter group, i.e., $\theta U_t \theta = U_{-t}$ and $U_t\mathcal{E}_+ \subseteq \mathcal{E}_+$ for $t \geq 0$, then we obtain the one-parameter semigroup $\widehat{U_t} = e^{-tH}$ of symmetric contractions on $\widehat{\mathcal{E}}$ such that

$$JU_tJ = U_t \quad \text{and} \quad \widehat{J}U^c_t\widehat{J} = U^c_{-t}.$$

For the triple $(\mathbb{T} \cong \mathbb{R}/\mathbb{Z}, \mathbb{T}^+, \tau)$ one expects the opposite relations

$$JU_tJ = U_{-t} \quad \text{and} \quad \widehat{J}U^c_t\widehat{J} = U^c_t, \qquad t \in \mathbb{R}.$$

To proceed beyond one-parameter groups, recall that in Borchers' theory of modular inclusions [Bo92], one encounters modular pairs (J, Δ) and pairs of unitary one-parameter groups (U, U') satisfying

$$\Delta^{it}U(s)\Delta^{-it} = U(e^{-2\pi t}s) \quad \text{and} \quad JU(s)J = U(-s)$$
$$\Delta^{it}U'(s)\Delta^{-it} = U'(e^{2\pi t}s) \quad \text{and} \quad JU'(s)J = U'(-s).$$

We thus obtain a unitary representation

$$\pi^c(s_1, s_2, t) = U(s_1)U'(s_2)\Delta^{-itx}$$

of the Poincaré group $\mathbb{R}^2 \rtimes \mathrm{SO}_{1,1}(\mathbb{R})$ in dimension 2.

In this context one would like to find natural euclidean realizations of such representations by representations of the euclidean motion group $\mathbb{R}^2 \rtimes \mathrm{O}_2(\mathbb{R})$, where the reflections in $\mathrm{O}_2(\mathbb{R})$ act by antiunitary involutions. Such euclidean realizations should connect naturally with the euclidean field theorie studied by Schlingemann [Sch99].

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Reflection positive representations and their integration

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(joint work with S. Merigon, K-H. Neeb)

We present some new results on reflection positive representations and integration of Lie algebra representations. For details and more information see [MNÓ14]. This is a part of a long time project with K.-H. Neeb on reflection positivity, [NÓ13, NÓ14, NÓ15]. See also [JÓ98, JÓ00]. Let us start with some basic concepts and notation. For that see also the report by K.-H. Neeb [N14] in this volume. A *reflection positive Hilbert space* is a a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$ such that $\mathcal{E}$ is a Hilbert space $\mathcal{E}_+$ is a subspace and $\theta : \mathcal{E} \to \mathcal{E}$ is an involution. The positivity condition is $(u, u)_\theta = (\theta u, u) \geq 0$ for all $u \in \mathcal{E}$. Let $\mathcal{N} := \{u \in \mathcal{E}_+ \mid \|u\|_\theta = 0\}$ and denote by $\widehat{\mathcal{E}}$ the completion of $\mathcal{E}_+/\mathcal{N}$ in the norm $\| \bullet \|_\theta$. For densely defined maps $T : \mathcal{E}_+ \to \mathcal{E}_+$ with $(\mathcal{N}) \subseteq \mathcal{H}$, define $\hat{T} : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}$. Denote the adjoint with respect to $(\bullet, \bullet)_\theta$ by *.

In the following (G, H) will always be a symmetric pair with respect to the involution τ and $(\mathcal{E}, \mathcal{E}_+, \theta)$ will be a reflection positive Hilbert space. We recall that a unitary representation of G on $\mathcal{E}$ is *reflection positive* if

(RP1) $\theta\pi\theta = \pi \circ \tau$.

(RP2) $\pi(H)\mathcal{E}_+ = \mathcal{E}_+$.

(RP3) There exists a subspace $\mathcal{D} \subseteq \mathcal{E}_+ \cap \mathcal{E}^\infty$ such that $\widehat{\mathcal{D}}$ is dense in $\widehat{\mathcal{E}}$ and $d\pi(X)\mathcal{D} \subset \mathcal{D}$ for all $X \in \mathfrak{g}$. (In application it is often enough to assume that this holds for $X \in \mathfrak{q}$.)

Here $\mathrm{d}\pi$ is the derived representation, sometimes denoted by $d\pi$ or π^∞.

If π is reflection positive, then $h \mapsto \widehat{\pi(h)}$, $h \in H$, defines a unitary representation π_H of H. On the other hand $\mathrm{d}\pi(X) \subseteq \mathrm{d}\pi(X)^\star$ for $X \in \mathfrak{q}$. Finally, $\mathrm{d}\pi^c(X+iY) := \mathrm{d}\pi(X) + i\mathrm{d}\pi(Y)$, $X \in \mathfrak{h}, Y \in \mathfrak{q}$, defines a (formally) infinitesimal unitary representation of $\mathfrak{g}^c$ such that, again formally as we have not assumed that $\pi_H(H)\widehat{\mathcal{D}} = \widehat{\mathcal{D}}$, $\pi_H(h)\mathrm{d}\pi^c(Z)\pi_H(h^{-1}) = \mathrm{d}\pi^c(\mathrm{Ad}(h)Z)$, $h \in H, Z \in \mathfrak{g}^c$. The question is then, if one can integrate $\mathrm{d}\pi^c$ to a unitary representation of G^c. By that we mean, if there exists a unitary representation $\widehat{\pi}$ of G^c on $\widehat{\mathcal{E}}$ such that $\mathrm{d}\widehat{\pi}(Z)|_{\widehat{\mathcal{D}}} = \mathrm{d}\pi^c(Z)$ for all $Z \in \mathfrak{g}^c$.

The first few articles on this problems include [FOS83, J86, J87, LM75, S86]. We will not discuss those articles here, just few words about the Lüscher-Mack Theorem [LM75]. For that one assumes that there exists a H-invariant open convex cone $C \subseteq \mathfrak{q}$ such that $S := H \exp C \simeq H \times C$ is a open semigroup in G invariant under $s^\sharp = \tau(s)^{-1}$. Assume further that $\pi(S)\mathcal{E}_+ \subseteq \mathcal{E}_+$. Then $\widehat{\pi(s)} : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}$ is well defined and $\widehat{\pi(s)}^\star = \widehat{\pi(s^\sharp)}$. It follows that $\widehat{\pi(\exp X)}$, $X \in \mathfrak{q}$, is a symmetric contraction on $\widehat{\mathcal{E}}$. In particular, the infinitesimal generator is self-adjoint with negative spectrum. The conclusion of the Lüscher-Mack theorem is that $\widehat{\pi}$ always exists.

But the spectral condition implies that $\widehat{\pi}$ restricted to the analytic subgroup of G^c corresponding to the Lie algebra generated by $i\mathfrak{q}$ is a direct integral of lowest weight representations, a condition that puts a very restrictive condition on $\mathfrak{g}$ and π. In particular if G is simple it implies that G^c/K^c, K^c maximal compact in G^c, has to be a bounded symmetric complex domain and the theorem works then well for certain classes of representations for the automorphism group tube type domains as discussed in [JÓ98, JÓ00]. On the other hand, it is not valid for the Heisenberg group nor the euclidean motion group.

The key idea in [MNÓ14] is the interplay between geometry, group and Lie algebra action and positive definite kernels. It is achieved by realizing the representation (π^c, π_H) in a geometric setting which is rich enough to imply integrability to a representation of G^c. For that one consider Hilbert spaces $\mathcal{H} = \mathcal{H}_K$ defined by a smooth reproducing kernel K on a locally convex manifold M and which are compatible with a *smooth action* (β, σ) of $(\mathfrak{g}, H)$, which means that $\sigma : M \times H \to M$ is a smooth right action and $\beta : \mathfrak{g} \to \mathcal{V}(M)$ is a homomorphism of Lie algebras for which the map $\hat{\beta} : \mathfrak{g} \times M \to TM, (X, m) \mapsto \beta(X)(m)$ is smooth, $\mathrm{d}\sigma(X) = \beta(X)$ for $X \in \mathfrak{h}$, and each vector field $\beta(X)$, $X \in \mathfrak{q}$, is locally integrable. The compatibility between K and (β, σ) can be expressed by $\mathcal{L}^1_{\beta(X)} K = -\mathcal{L}^2_{\beta(\tau X)} K$ for $X \in \mathfrak{g}$. Recall that $\mathcal{H}_K$ is the completion of

$$\mathcal{H}_K^o = \left\{ \sum_{\text{finite}} c_j K_{x_j} \;\middle|\; c_j \in \mathbb{C}, x_j \in M \right\}$$

with $K_y(x) = K(x, y)$ and inner product determined by $(K_x, K_y) = K(y, x)$.

We prove a geometric version of Fröhlich's Theorem on selfadjoint operators and corresponding local linear semiflows [F80]:

Theorem[Geometric Fröhlich Theorem] *Let M be a locally convex manifold and K a smooth positive definite kernel. If X is a vector field such that on M such that $\mathcal{L}_{X,x}K(x,y) = \mathcal{L}_{X,y}K(x,y)$ and for all $m \in M$ there exists an integral curve $\gamma_m : [0,\epsilon_m] \to M$, $\gamma_m(0) = m$, then $\mathcal{L}_X|_{\mathcal{H}_K^o}$ is essential self-adjoint and its closure $\overline{\mathcal{L}_K}$ satisfies*

$$e^{t\overline{\mathcal{L}_K}}K_m = K_{\gamma_m(t)}\,.$$

Using this theorem one shows, that a $(\mathfrak{g}, H)$-compatible action as above leads to a representation of $\mathfrak{g}_{\mathbb{C}}$ on a dense domain $\mathcal{D}$ in the corresponding reproducing kernel Hilbert space $\mathcal{H}_K$, with $\mathcal{H}_K^o \subseteq C^\infty(M)$. Furthermore, $\mathfrak{g}^c$ acts by essentially skew-adjoint operator. That one can integrate the representation relies on a result in [M11].

We apply this general result to several situations, one of them being generalization (and simplification) to Banach Lie groups of Jorgensen's theorem on integration of local representations. Another interesting example is the situation where $\emptyset \neq U \subseteq G$ is open, and $UH = U$. We say that $\varphi : UU^\sharp \to \mathbb{C}$ is τ-positive definite if the kernel $K(x,y) = \varphi(xy^\sharp)$ is positive definite. Define a $(\mathfrak{g}^c, H)$ action on $\mathcal{H}_K$ by $\pi^c(h)F(g) = F(gh)$ and $\mathrm{d}\pi^c(iY)F = i\mathcal{L}_Y F$, $Y \in \mathfrak{q}$. Then $\hat{\pi}$ exists. We get the Lüscher-Mack Theorem as a simple corollary by taking $U = S$.

Assume now that M is a finite dimensional manifold with a G-action. Assume that D is a positive definite distribution on $M \times M$ (assumed conjugate linear). Then $K_D(\varphi, \psi) = D(\overline{\varphi} \otimes \psi)$ defines a positive definite kernel on $C_c^\infty(M)$ and hence a Hilbert space $\mathcal{E} = \mathcal{H}_K$. Let $\tau_M : M \to M$ be an involution such that $\tau_M(g \cdot m) = \tau(g) \cdot \tau_M(m)$. Let $\theta F = F \circ \tau_M$. Finally let $M_+ \neq \emptyset$ be an open H-invariant sub-manifold and such that $D \circ (\tau_M, \mathrm{id})$ is positive definite on M_+. We let $\mathcal{E}_+$ be the image of $C_c^\infty(M_+)$ in $\mathcal{E}$. Then $(\mathcal{E}, \mathcal{E}_+, \theta)$ is reflection positive and we get a compatible $(\mathfrak{g}^c, H)$-action which integrates to a representation $\widehat{\pi}$ of G^c.

Here typical examples are obtained from reflection positive representations of (G, S, τ), where $M_+ = S \subseteq G$ is a $\sharp$-invariant open subsemigroup. If ν is a reflection positive distribution vector, see [NÓ13] for definition, then

$$K_\nu(f,g) = \langle \nu, \pi^{-\infty}(f^* * g)\nu \rangle \quad \text{for} \quad f,g \in C_c^\infty(G)$$

is a reflection positive distribution on G and the above results for distributions applies to get a representation $\widehat{\pi}$ of G^c. Note, that it is not required that, as in the Lüscher-Mack Theorem, the semigroup S has a polar decomposition.

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Integral Representations in Euclidean Quantum Field Theory

Jakob Yngvason

The talk was a review of some old results about representations of the Schwinger functions ("euclidean Green's functions") in quantum field theory as moments of measures on spaces of distributions. These results, obtained in the 1970's and 80's, are apparently not well known among mathematicians seeking new applications of reflection positivity, a concept originating in the seminal papers of Konrad Osterwalder and Robert Schrader from 1973–75 [1]. A further motivation for the talk was recent work in quantum field theory [2] where more general types of complex linear functionals than sigma-additive measures appear naturally.

The mathematical setting is as follows. The *Borchers-Uhlmann algebra* is the tensor algebra over Schwartz space of rapidly decreasing test functions $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$, $d \in \mathbb{N}$.

$$\underline{\mathcal{S}} = \bigoplus_{n=0}^{\infty} \mathcal{S}_n$$

with

$$\mathcal{S}_0 = \mathbb{C}, \quad \mathcal{S}_n = \mathcal{S}(\mathbb{R}^{nd}).$$

Its elements are sequences

$$\mathbf{f} = (f_0, \ldots, f_N, 0, \ldots), \quad f_n \in \mathcal{S}_n, \quad N < \infty,$$

addition and multiplication by scalars is defined component-wise, and the product is the tensor product:

$$(\mathbf{f} \otimes \mathbf{g})_n(x_1, \ldots, x_n) = \sum_{\nu=0}^{n} f_\nu(x_1, \ldots, x_\nu) g_{n-\nu}(x_{\nu+1}, \ldots, x_n).$$

Note that the n-fold tensorial power $\mathcal{S}(\mathbb{R}^d)^{\otimes n}$ of the basic test function space $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{S}_n$ in the Schwartz topology. There is also an antilinear involution:

$$(\mathbf{f}^*)_n(x_1, \ldots, x_n) = \overline{\mathbf{f}_n(x_n, \ldots, x_1)}.$$

With this structure $\underline{\mathcal{S}}$ is a topological $*$-algebra.

In euclidean quantum field theory one is concerned with the dual space $\underline{\mathcal{S}}'$, consisting of the continuous linear functionals on $\underline{\mathcal{S}}$, i.e.,

$$S = (S_0, S_1, \ldots) \in \underline{\mathcal{S}}', \quad S_n \in \mathcal{S}_n', \quad \text{with} \quad S(\mathbf{f}) = \sum_n S_n(f_n).$$

It is required that the tempered distributions S_n are *totally symmetric*, i.e., invariant under the natural action of the permutation group on $\mathcal{S}(\mathbb{R}^d)^{\otimes n}$. Moreover, they should be *euclidean invariant*, i.e., invariant under the natural action of the d-dimensional euclidean group on the test functions.

If $\mathcal{S}_{\mathrm{R}}'$ is the space of real, tempered distributions on $\mathbb{R}^d$ and $\omega \in \mathcal{S}_{\mathrm{R}}'$, there is a corresponding hermitian character χ_ω on the algebra $\underline{\mathcal{S}}$, defined by

$$(\chi_\omega)_n = \omega^{\otimes n}.$$

In [3, 4] the following questions were addressed:

Q1. When does a symmetric functional S have an integral representation

$$S = \int_{\mathcal{S}_{\mathrm{R}}'} \chi_\omega \, \mathrm{d}\mu(\omega)$$

with a positive, finite, sigma-additive measure $\mathrm{d}\mu(\omega)$ on the space $\mathcal{S}_{\mathrm{R}}'$ of real, tempered distributions on $\mathbb{R}^d$?

Q2. When is there at least such a representation by a signed or complex sigma-additive, finite measure?

These questions can be regarded as infinite dimensional versions of classical moment problems with $S_n = \int \omega^{\otimes n} \mathrm{d}\mu(\omega)$ corresponding to the n-th moment of $\mathrm{d}\mu(\omega)$. In [5] a further question is discussed:

Q3. When is there a representation as in Q1 or Q2 with a measure that is invariant under the action of the euclidean group on $\mathcal{S}_{\mathrm{R}}'$?

The answers are as follows:

A1. A representation by a positive measure is possible if and only if S is positive on all "positive polynomials", i.e, all $\mathbf{f} \in \underline{\mathcal{S}}$ such that $\omega \mapsto \chi_\omega(\mathbf{f})$ a nonnegative function on $\mathcal{S}_{\mathrm{R}}'$ [3].

A2. A representation by a signed or complex measure is possible if and only if there are Schwartz-norms $\| \cdot \|_k$, $k = 1, 2, \ldots$ on $\mathcal{S}(\mathbb{R}^d)$ such that

$$|S_n(g_1 \otimes \cdots \otimes g_n)| \leq \|g_1\|_1 \cdots \|g_n\|_n \tag{1}$$

for all $g_i \in \mathcal{S}(\mathbb{R}^d)$ [4].

A3. If S is euclidean invariant and has a representation by a positive measure, there is also a representation by a euclidean invariant positive measure.

If the S_n are the euclidean invariant Schwinger functions of a Wightman quantum field theory in the sense of [1] and there is a mass gap in the energy-momentum spectrum spectrum, then condition (1) also guarantees a representation by a euclidean invariant signed or complex measure.

The last statement is highly nontrivial as can be seen from the example of the time-ordered functions of a free Wightman field [5].

A signed or complex *gaussian measure* on $\mathcal{S}'_{\mathrm{R}}$ is a measure $\mathrm{d}\mu(\omega)$ such that

$$\int e^{\mathrm{i}\omega(f)}\mathrm{d}\mu(\omega) = e^{-Q(f)}$$

for $f \in \mathcal{S}_{\mathrm{R}}$, where Q is a continuous quadratic form on $\mathcal{S}_{\mathrm{R}}$ (in general complex valued). The moments $S_n = \int \omega^{\otimes n}\mathrm{d}\mu(\omega)$ of such a measure have a gaussian structure: $S_{2n+1} = 0$, and

$$S_{2n}(x_1, \ldots, x_{2n}) = \sum_{\text{partitions}} \prod_{i_k < j_k} S_2(x_{i_k}, x_{j_k}). \tag{2}$$

It was shown in [5] that the following are equivalent:

- The S_n are moments of a complex, gaussian measure.
- Re $S_2(f \otimes f) \geq 0$ for all $f \in \mathcal{S}_{\mathrm{R}}$ and Im $S_2(f \otimes g)$ is given by a Hilbert Schmidt operator w.r.t. the scalar product $(f, g) \mapsto \operatorname{Re} R_2(f \otimes g)$, $f, g \in \mathcal{S}_{\mathrm{R}}$.

On the other hand, if (2) holds, then condition (1) is clearly fulfilled. Thus, by A2 there is a representation by *some* signed or complex measure.

Complex measures such that all their moments exist and are tempered distributions can be regarded as the dual space of the algebra $\mathcal{F}$ of "polynomially bounded functions" on $\mathcal{S}'_{\mathrm{R}}$. These are functions of the form

$$F(\omega) = f(\chi_\omega(\mathbf{f}_1), \ldots, \chi_\omega(\mathbf{f}_n)) \tag{3}$$

with $\mathbf{f}_i \in \underline{\mathcal{S}}$ and $f : \mathbb{C}^n \to \mathbb{C}$ polynomially bounded and continuous. There is a natural topology on $\mathcal{F}$ given by seminorms of the form

$$\sup_\omega \frac{|F(\omega)|}{\|\chi_\omega\|^0}$$

where $\|\cdot\|^0$ is the dual seminorm on $\underline{\mathcal{S}}'$ of a continuous seminorm $\|\cdot\|$ on $\underline{\mathcal{S}}$. The complex measures required in Q1 correspond precisely to linear functionals on $\mathcal{F}$ that are continuous in this topology. The condition (1) results from restricting this topology to functions of the form (3) with f a polynomial.

Selecting one coordinate in $\mathbb{R}^d$ as a "time" coordinate we denote by $\mathcal{F}_+$ the subalgebra generated by functions of the form (3) where the supports of the components of the $\mathbf{f}_i$ are in the half-space where the time coordinate is non-negative. *Reflection positivity* for a measure $\mathrm{d}\mu(\omega)$ is the condition

$$\int \overline{\theta F(\omega)} F(\omega)\mathrm{d}\mu(\omega) \geq 0$$

for all $F \in \mathcal{F}_+$, where θ is time reflection.

The following somewhat surprising result was proved in [6]:

Theorem. *If* $\mathrm{d}\mu$ *is invariant under time translations and reflection positive, then* $\mathrm{d}\mu$ *is a positive measure.*

The proof relies essentially on the polar decomposition of the measure, $\mathrm{d}\mu(\omega) = e^{\mathrm{i}\alpha(\omega)}\mathrm{d}|\mu|(\omega)$ with $\mathrm{d}|\mu|(\omega)$ a bounded positive measure and $\alpha(\cdot)$ real valued and bounded. It does not exclude reflection positive but non-positive complex functionals that are not given by a sigma additive meansure.

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Semibounded covariant representations of the Weyl relations

Christoph Zellner

Let (V, ω) be a locally convex symplectic space and $\mathrm{Heis}(V, \omega) = \mathbb{R} \times_\omega V$ the Heisenberg group. Suppose in addition we are given a one-parameter group $\gamma : \mathbb{R} \to \mathrm{Sp}(V, \omega)$ defining a smooth action of $\mathbb{R}$ on V. A continuous unitary representation $\pi : \mathrm{Heis}(V, \omega) \to \mathrm{U}(\mathcal{H})$ with $\pi(t, 0, 0) = e^{it}\mathbf{1}$ satisfies the Weyl relations

$$\pi(x)\pi(y) = e^{\frac{1}{2}\omega(x,y)}\pi(x + y), \quad x, y \in V.$$

We are interested in such representations π which admit an implementation of γ, i.e., a unitary one-parameter group U_t on $\mathcal{H}$ satisfying $U_t\pi(x)U_{-t} = \pi(\gamma(t)x)$, such that the self-adjoint generator of U_t is bounded below. Note that the implementation of γ corresponds to an extension of π to a representation of a so-called *oscillator group* $G(V, \omega, \gamma) := \mathrm{Heis}(V, \omega) \rtimes_\gamma \mathbb{R}$. More precisely we want to report on semibounded representations of $G(V, \omega, \gamma)$: Let $\pi : G \to \mathrm{U}(\mathcal{H})$ be a smooth unitary representation of a locally convex Lie group G and $\mathtt{d}\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ its derived representation. Then π is called *semibounded* if the self-adjoint operators $\overline{i\mathtt{d}\pi(x)}$ are uniformly bounded above for x in a non-empty open subset of the Lie algebra $\mathfrak{g}$ ([3]). In this case we set $B_\pi := \{x \in \mathfrak{g} : \overline{i\mathtt{d}\pi(x)} \text{ is bounded above}\}^0$.

A *standard oscillator group* is of the form $G_A = \mathrm{Heis}(V_A, \omega_A) \rtimes_\gamma \mathbb{R}$, where γ is a unitary one-parameter group on a Hilbert space H, $V_A := C^\infty(A)$ is the space of γ-smooth vectors equipped with its natural C^∞-topology, $\omega_A(x, y) = \mathrm{Im}\langle Ax, y\rangle$ and A is the self-adjoint generator of γ satisfying $A \geq 0$ and $\ker A = \{0\}$. The following theorem clarifies the importance of standard oscillator groups when considering semibounded representations.

Theorem 1. *Let $G := G(V, \omega, \gamma)$ be an oscillator group.*

(a) *The following are equivalent:*
   (i) $\exists\ \pi : G \to \mathrm{U}(\mathcal{H})$ *semibounded with* $\pi(t, 0, 0) = e^{it}\mathbf{1}$.
   (ii) $\exists$ *standard oscillator group* G_A *and a dense continuous inclusion* $\iota : V \hookrightarrow C^\infty(A)$ *such that* $\iota : G \hookrightarrow G_A, (t, v, s) \mapsto (t, \iota(v), s)$ *is a morphism of Lie groups (after possibly replacing* γ *by* γ^{-1}*).*

(b) *Assume (a),* $\gamma(\mathbb{R}) \subset \mathrm{End}(V)$ *equicontinuous and* $DV \subset V$ *dense, where* $D := \gamma'(0)$*. Then every semibounded representation* π *of* G *extends to a (unique) semibounded representation* $\hat{\pi}$ *of* G_A*, where* G_A *is as in (a).*

For a standard oscillator group G_A a smooth representation $\pi : G_A \to \mathrm{U}(\mathcal{H})$ with $\pi(t, 0, 0) = e^{it}\mathbf{1}$ is semibounded if and only if $-i\mathtt{d}\pi(0, 0, 1)$ is bounded below. However for a general oscillator group $G(V, \omega, \gamma)$ this equivalence does not always hold. Every G_A has a natural semibounded irreducible representation the so-called *Fock representation* $\pi_F : G_A \to \mathrm{U}(\mathcal{H}_F)$ defined by the positive definite function $\varphi(t, x, s) = e^{it - \frac{1}{4}\langle Ax, x\rangle}$. The following theorem determines the semibounded representations of G_A in the case $\inf \mathrm{Spec}(A) > 0$.

Theorem 2. *Assume* $\inf \mathrm{Spec}(A) > 0$ *and let* $\pi : G_A \to \mathrm{U}(\mathcal{H})$ *be semibounded with* $\pi(t, 0, 0) = e^{it}\mathbf{1}$*. Then*

$$(\pi, \mathcal{H}) \cong (\pi_F \otimes \nu, \mathcal{H}_F \otimes \mathcal{K})$$

where $\nu(t, x, s) = \tilde{\nu}(s)$ *and* $\tilde{\nu} : \mathbb{R} \to \mathrm{U}(\mathcal{K})$ *is a one-parameter group with self-adjoint generator bounded below.*

In particular, $\pi|_{\mathrm{Heis}(V_A)}$ *is equivalent to a direct sum of Fock representations* $\pi_F|_{\mathrm{Heis}(V_A)}$*. If* π *is irreducible then* $\dim \mathcal{K} = 1$.

The proof of this theorem requires a careful analysis of the space of smooth vectors and related techniques, as discussed in [4] in a more general setting. In the case $A = \mathrm{Id}$ the preceding result was already obtained by Chaiken [1].

The next result shows that the space of smooth vectors for a semibounded representation π of G_A with $\pi(t, 0, 0) = e^{it}\mathbf{1}$ is determined by the single one-parameter group $t \mapsto \pi(\exp(tx))$ whenever $x \in B_\pi$.

Theorem 3. *Let* $\pi : G_A \to \mathrm{U}(\mathcal{H})$ *be semibounded with* $\pi(t, 0, 0) = e^{it}\mathbf{1}$*. Then* $\mathcal{H}^\infty = \mathcal{D}^\infty(i\mathtt{d}\pi(x))$ *for all* $x \in B_\pi = \mathbb{R} \times V_A \times]0, \infty[$*. Moreover*

$$\|\mathtt{d}\pi(x)v\|^2 \leq 4\|x\|^2\langle Hv, v\rangle + 2\langle Ax, x\rangle \cdot \|v\|^2$$

for all $x \in V_A, v \in \mathcal{H}^\infty$ *if* $H := \frac{1}{i}\mathtt{d}\pi(0, 0, 1) \geq 0$.

This theorem implies that $e^{i\mathrm{d}\pi(x)}$ is a smoothing operator for $x \in B_\pi$ in the sense that $e^{i\mathrm{d}\pi(x)}\mathcal{H} \subset \mathcal{H}^\infty$. By considering the C^*-algebra $C^*(\pi(G_A)e^{i\mathrm{d}\pi(B_\pi)}\pi(G_A)) \subset B(\mathcal{H})$ and using results of Neeb and Salmasian on smoothing operators it follows that every semibounded representation $\pi : G_A \to \mathrm{U}(\mathcal{H})$ is a direct integral of semibounded factor representations if G_A and $\mathcal{H}$ are separable. This was also obtained in [6] using different techniques.

In the following let G_A be a standard oscillator group and assume V_A is separable, $\dim(V_A) = \infty$ and A is diagonalizable, i.e., $Ae_k = a_k e_k$ with $a_k > 0$ for an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ in V_A. Let $V^0 = \oplus_k \mathbb{C}e_k$ denote the algebraic span of the $e_k, k \in \mathbb{N}$. A representation $\pi : \mathrm{Heis}(V^0, \omega_A) \to \mathrm{U}(\mathcal{H})$ is called *regular* if $\pi(t, 0) = e^{it}\mathbf{1}$ and π is continuous on one-parameter groups (ray continuous). In [2] a description of all regular representations of $\mathrm{Heis}(V^0, \omega_A)$ was obtained in terms of tuples $(\mu, \nu, c_k(n))$. Here $\mathcal{H} \cong \int_{\mathbb{N}_0^\infty} \mathcal{H}_n d\mu(n)$ where μ is a probability measure on $\mathbb{N}_0^\infty, \nu(n) = \dim \mathcal{H}_n$ and $c_k(n) : \mathcal{H}_{n+\delta_k} \to \mathcal{H}_n$ is a measurable field of unitaries satisfying some conditions, cf. [2] for the details. With the help of Theorem 3 the following can be obtained.

Theorem 4. *Let $\pi : \mathrm{Heis}(V^0, \omega_A) \to \mathrm{U}(\mathcal{H})$ be a regular representation. Then π extends to a semibounded representation $\hat{\pi} : G_A \to \mathrm{U}(\mathcal{H})$ if and only if*

$$\psi : \mathbb{N}_0^\infty \to \mathbb{R} \cup \{\infty\}, (n_k)_k \mapsto \sum_k a_k n_k$$

is finite μ-almost everywhere.

This yields a description of all semibounded representations π of G_A with $\pi(t, 0, 0) = e^{it}\mathbf{1}$ (up to equivalence) in terms of tuples $(\mu, \nu, (c_k)_{k\in\mathbb{N}}, \eta)$ given as follows: The tuple $(\mu, \nu, (c_k)_{k\in\mathbb{N}})$ is as in [2], μ satisfies the additional condition that ψ is finite μ-almost everywhere and $\eta : \mathbb{R} \to \mathrm{U}(\mathcal{H}) \cap \pi_h(\mathrm{Heis}(V^0))'$ is a continuous unitary one-parameter group with self-adjoint generator bounded below, where π_h denotes the representation of $\mathrm{Heis}(V^0)$ on $\mathcal{H}$ corresponding to $(\mu, \nu, (c_k)_k)$.

Most of the results presented in this report can be found in [6].

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Problem Sessions

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To get an overview over the current problems concerning reflection positivity (RP), four discussion sessions were organized during the second half of the workshop. The list of problems that were discussed during those sessions is:

(1) Describe super versions of massive free fields.
(2) What is the significance of anti-unitary involutions play in RP?
(3) Do Borchers triples have natural euclidean realizations?
(4) What is the natural super version of a reflection positive Hilbert space?
(5) Is it possible to construct a euclidean version of algebraic QFTs?
(6) Find more examples of diffeomorphism invariant reflection positive QFTs.
(7) Is there a notion of reflection positivity in non-commutative geometry?
(8) Clarify the connection between RP and complex-valued measures?
(9) Is there a variant of the Lax–Philips Representation Theorem for one-parameter groups acting on standard real subspaces?
(10) How to relate stochastic quantization with reflection positivity?
(11) What are interesting situations where RP occurs without a euclidean Hilbert space?
(12) What is the common core of various aspects of reflection positivity?
(13) Is it possible to connect reflection positivity and number theory?
(14) Describe the duality properties related to multiple involutions with RP.
(15) What is the role of reflection positivity in String Theory?
(16) Which representations arise from reflection positivity in 1d CFT?
(17) Show the equivalence of the Osterwalder–Schrader and the Wightman axioms without growth estimates, resp., regularity assumptions.
(18) Describe the spectrum for multiplarticle cluster expansions compatible with reflection positivity beyond the mass gap.
(19) Is it possible to relate the modular theory of von Neumann algebras directly to physical effects?
(20) Are there connections between the BMV Theorem and RP?

During the discussion session the participants had a lively exchange on these questions. Some of them are deep open problems, others rather ask for background information that could be provided by the participants. The following comments reflect some of the main points of the discussions.

Ad (1), (4): Presently we don't have a description of reflection positivity in terms of the recently developed theory of unitary representations of Lie supergroups (cf. [CCTV06], [NS13]). Here, the fermionic fields studied by Osterwalder–Schrader [OS73, OS75] form a natural starting point (see also [Va04] for super Poincaré groups and super space times). A closely related problem is to generalize the representation theoretic side of reflection positivity ([NO14a, NO14b]) and in particular the concept of a reflection positive Hilbert space to the super context.

Ad (2): So far, antiunitary involutions have not been discussed systematically on the representation theoretic side of reflection positivity, but they naturally enter the scene when the underlying groups are compact ([NO15]; see also [Ne15] which in turn connects to modular theory of operator algebras).

Ad (3), (5): In [Sch99] Schlingenmann develops a euclidean version of Algebraic Quantum Field Theory for a euclidean invariant system of C^*-algebras $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \subseteq \mathbb{R}^d$. This includes a reflection positivity requirement $\omega(\theta(A^*)A) \geq 0$ for the vacuum state on the subalgebra $\mathcal{A}(\mathbb{R}^d_+)$. What is presently not clear is how to find natural euclidean realizations of a Poincaré invariant QFT. The term "Borchers triple" refers to a triple $(\mathcal{M}, U, \Omega)$, where $\mathcal{M}$ is a von Neumann algebra of operator on a Hilbert space $\mathcal{H}$, Ω a faithful separating state of $\mathcal{M}$ and U is a unitary representation of the group of space-time translations fixing Ω, with spectrum supported in the forward light cone, and where conjugation with the elements from a wedge domain W map $\mathcal{M}$ into itself ([Bo92]). Since they are key ingredients in QFTs, a basic problem is to find natural euclidean realizations for Borchers triples in the sense of [Ne15].

Ad (6): This questions asks for variations of the QFTs constructed by Ashtekar, Lewandowski, Marolf, Mourao and Thiemann in [ALMMT97, AMMT00].

Ad (7): To find a answer to this question is part of an ongoing project of Grosse and Wulkenhaar; see [GW13, GW14] for recent progress.

Ad (8), (17): In the work of Osterwalder–Schrader [OS73, OS75, GJ81], the equivalence between the Wightman axioms and the OS axioms is shown under the assumptions that the Schwinger and the Wightman "functions" S_n and W_n define tempered distributions satisfying some growth condition in n. Since there exist Wightman fields violating these regularity conditions, it would be of some interest to see if they also correspond to less regular Schwinger "functions". This is closely related to the approach to QFTs where Wightman distributions are interpreted as a positive functional on the Borchers–Uhlmann (BU) algebra ([Bo62]). In this context complex measures naturally enter the picture by representating the linear functionals on the BU algebra ([BY76]); for connections to reflection positivity, we refer to [JRM13]. Here a central issue is to understand what the natural test function spaces should be.

Ad (9): This refers to Lax–Phillips scattering theory in its abstract form for unitary one-parameter groups in Hilbert spaces which satisfy certain axioms and for which is provides a unique normal form for such one-parameter groups. This problem asks for an extension of this normal form result to for one-parameter groups $(U_t)_{t\in\mathbb{R}}$ whose positive part leaves a standard real subspace $V \subseteq \mathcal{H}$ invariant, i.e., $V \cap iV = \{0\}$ and $V + iV$ is dense. These situations arise from Borchers triples in modular theory.

Ad (19): Modular theory, also called Tomita–Takesaki theory, lies at the foundation of rigorous formulations of equilibrium states in statistical mechanics and Borchers' formulation of local quantum fields ([Bo92]) makes critically use of it. Here a key result is the Bisognano-Wichmann Theorem which roughly asserts that for an algebraic QFT and the canonical right wedge W in 4-dimensional Minkowski space, the Tomita–Takesaki modular objects (J_W, Δ_W) can be expressed in terms of the PCT operator and the action of the Poincaré group (cf. [Yn94]).

Ad (20): In 2013, Stahl proved the conjecture (due to Bessis, Moussa and Villani) which asserts that, for two hermitian $n \times n$-matrices A, B the function $\mathrm{tr}(e^{A-tB})$ is a Laplace transform of a positive measure on the interval between the minimal and maximal eigenvalues of B (cf. [Er13], [St12]).

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